ADDITIVE STRUCTURES IN SUMSETS

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Abstract. Suppose that A and A' are subsets of $\mathbb{Z}/N\mathbb{Z}$. We write $A+A'$ for the set $\{a+a' : a \in A \text{ and } a' \in A'\}$ and call it the *sumset* of A and A'. In this paper we address the following question. Suppose that $A_1, ..., A_m$ are subsets of $\mathbb{Z}/N\mathbb{Z}$. Does $A_1 + ... + A_m$ contain a long arithmetic progression?

The situation for $m = 2$ is rather different from that for $m \geq 3$. In the former case we provide a new proof of a result due to Green. He proved that $A_1 + A_2$ contains an arithmetic progression of length roughly $\exp(c\sqrt{\alpha_1\alpha_2 \log N})$ where α_1 and α_2 are the respective densities of A_1 and A_2. In the latter case we improve the existing estimates. For example we show that if $A \subset \mathbb{Z}/N\mathbb{Z}$ has density $\alpha \gg \sqrt{\log\log N/\log N}$ then $A+A+A$ contains an arithmetic progression of length $N^{c\alpha}$. This compares with the previous best of $N^{c\alpha^{2+\varepsilon}}$.

Two main ingredients have gone into the paper. The first is the observation that one can apply the iterative method to these problems using some machinery of Bourgain. The second is that we can localize a result due to Chang regarding the large spectrum of L^2-functions. This localization seems to be of interest in its own right and has already found one application elsewhere.

1. Introduction

As indicated in the abstract we are interested in the following question.

Question 1.1. *Suppose that $A_1, ..., A_m \subset \mathbb{Z}/N\mathbb{Z}$. Does $A_1 + ... + A_m$ contain a long arithmetic progression?*

The case $m = 2$ is much harder than $m \geq 3$; the best known bounds lie with Green, [Gre02a], who proved the following result.

Theorem 1.2. *Suppose that $A_1, A_2 \subset \mathbb{Z}/N\mathbb{Z}$. Suppose that α, the geometric mean of the densities of the A_is, is positive. Then A_1+A_2 contains an arithmetic progression of length at least $\exp(c((\alpha^2 \log N)^{\frac{1}{2}} - \log\log N))$ for some absolute constant $c > 0$.*

The next result is proved for $m = 3$ and $A_1 = A_2 = A_3$ in [Gre02a], although the more general conclusion can easily be read out of that paper. The special case $m = 4$, $A_1 = A_2 = A$, $A_3 = A_4 = -A$ can be read out of earlier work of Chang [Cha02].

Theorem 1.3. *Suppose that $m \geq 3$ and $A_1, ..., A_m \subset \mathbb{Z}/N\mathbb{Z}$. Suppose that α, the geometric mean of the densities of the A_is, is positive. Then $A_1 + ... + A_m$ contains an arithmetic progression of length at least $c\alpha^C N^{cm^{-1}\alpha^{\frac{2}{m-2}}(\log \alpha^{-1})^{-1}}$ for some absolute constants $C, c > 0$.*

In this paper we prove the following two results.

Theorem 1.4. *Suppose that $A_1, A_2 \subset \mathbb{Z}/N\mathbb{Z}$. Suppose that α, the geometric mean of the densities of the A_is, is positive. Then $A_1 + A_2$ contains an arithmetic progression of length at least $\exp c(((\alpha^2 \log N)^{\frac{1}{2}} - \log \alpha^{-1} \log\log N)$ for some absolute constant $c > 0$.*

Theorem 1.5. *Suppose that $m \geq 3$ and $A_1, ..., A_m \subset \mathbb{Z}/N\mathbb{Z}$. Write α for the geometric mean of the densities of the A_is. Then $A_1 + ... + A_m$ contains an arithmetic progression of length $c\alpha^{Cm^3\alpha^{-\frac{1}{m-2}}} N^{cm^{-2}\alpha^{\frac{1}{m-2}}}$ for some absolute constants $C, c > 0$.*

Theorem 1.5 is stronger than Theorem 1.3, while Theorem 1.4 is marginally weaker than Theorem 1.2. Despite this, we believe that merit can still be found in our approach to Theorem 1.4 for two reasons:

- The real strength of Theorems 1.2 and 1.4 is when A_1 and A_2 are thick sets, and in that case the slightly weaker error term in Theorem 1.4 plays no part. The case when A_1 and A_2 are thin is addressed by Croot, Ruzsa and Schoen in [CRS07].
- Green's proof of Theorem 1.2 is a tour de force combining a number of powerful analytic tools in a highly non-trivial way. By contrast our method is conceptually simpler although probably technically more challenging.

In any case the merit of our approach can, perhaps, be most easily seen in the finite field setting where the two methods give the same result and the technicalities in our approach disappear to leave a fairly simple argument.

To put our refinement of Theorem 1.3 in context take, for example, $m = 3$ and $A_1 = A_2 = A_3 = A$. Theorem 1.5 is then equivalent to the fact that there are absolute constants $C, c > 0$ such that if

$$\alpha \geq C\sqrt{\frac{\log\log N}{\log N}} \text{ then } A + A + A \text{ contains a progression of length } N^{c\alpha}.$$

The previous best is essentially equivalent to the existence of absolute constants $C, c > 0$ such that if

$$\alpha \geq C\sqrt{\frac{(\log\log N)^2}{\log N}} \text{ then } A + A + A \text{ contains a progression of length } N^{c\alpha^{2+\varepsilon}}.$$

Here, of course, $\alpha^{2+\varepsilon}$ is shorthand for α^2 up to some logarithmic factors.

It is also the case, as we shall see in the next section, that the proof behind Theorem 1.5 gives stronger structural information than Theorem 1.3.

2. The Fourier transform, Bohr neighborhoods and additive structure

In this section we develop the results behind Theorems 1.4 and 1.5, and identify some of the mathematics necessary to prove them. Our main tool is the Fourier transform; we take a moment to set our notation in this regard.

Suppose that G is a compact Abelian group. We write $\widehat{G}$ for the dual group, that is the discrete Abelian group of continuous homomorphisms $\gamma : G \to S^1$, where $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. Although the natural group operation on $\widehat{G}$ corresponds to pointwise multiplication of characters we shall denote it by '+' in alignment with contemporary work. G may be endowed with Haar measure μ_G normalized

so that $\mu_G(G) = 1$ and as a consequence we may define the Fourier transform $\widehat{\cdot}: L^1(G) \to \ell^\infty(\widehat{G})$ which takes $f \in L^1(G)$ to

$$\widehat{f} : \widehat{G} \to \mathbb{C}; \gamma \mapsto \int_{x \in G} f(x)\overline{\gamma(x)} d\mu_G(x).$$

We can define a natural valuation on S^1, namely, given $z \in S^1$ we let $\|z\| := |\theta|$ where θ is the unique element of the interval $(-1/2, 1/2]$ such that $x = \exp(2\pi i \theta)$. This valuation can be used to measure how far $\gamma(x)$ is from 1. Suppose that $\Gamma \subset \widehat{G}$ and $\delta \in (0, 1]$. We define

$$B(\Gamma, \delta) := \{x \in G : \|\gamma(x)\| \leq \delta \text{ for all } \gamma \in \Gamma\},$$

and call such a set a *Bohr set* and a translate of such a set a *Bohr neighborhood*. The following is an easy pigeonhole argument and gives an estimate for the volume of these sets. See Lemma 4.20 in [TV06] for the details.

Lemma 2.1. *Suppose G is a compact Abelian group, Γ a set of d characters on G and $\delta \in (0, 1]$. Then $\mu_G(B(\Gamma, \delta)) \geq \delta^d$.*

Consequently we can write $\beta_{\Gamma,\delta}$, or simply β or β_δ if the parameters are implicit, for the measure induced on $B(\Gamma, \delta)$ by μ_G and normalized so that $\|\beta_{\Gamma,\delta}\|_1 = 1$. This is sometimes referred to as the *normalized Bohr cutoff*. We write β' for $\beta_{\Gamma',\delta'}$, or $\beta_{\Gamma,\delta'}$ if no Γ' has been defined. We have a similar convention for β''.

In §3 we shall see that Bohr sets in fact behave as sort of approximate groups and in particular they are highly additively structured. When G is a cyclic group this translates to Bohr sets containing long arithmetic progressions; specifically the following is another easy application of the pigeonhole principle. Again, see [TV06] for details.

Lemma 2.2. *Suppose that $G = \mathbb{Z}/N\mathbb{Z}$. Suppose that $B(\Gamma, \delta)$ is a Bohr set with $|\Gamma| = d$. Then $B(\Gamma, \delta)$ contains an arithmetic progression of length $\delta N^{\frac{1}{d}}$.*

Theorem 1.3 follows immediately from Lemma 2.2 and the following result about Bohr neighborhoods.

Theorem 2.3. *Suppose that G is a compact Abelian group. Suppose that $m \geq 3$ and $A_1, ..., A_m \subset G$. Suppose that α, the geometric mean of the densities of the A_is, is positive. Then $A_1 + ... + A_m$ contains a translate of a Bohr set $B(\Gamma, \delta)$ with*

$$|\Gamma| \ll m\alpha^{-\frac{2}{m-2}} \log \alpha^{-1} \text{ and } \log \delta^{-1} \ll \log \alpha^{-1}.$$

In this paper we prove the following refinement.

Theorem 2.4. *Suppose that G is a compact Abelian group. Suppose that $m \geq 3$ and $A_1, ..., A_m \subset G$. Suppose that α, the geometric mean of the densities of the A_is, is positive. Then $A_1 + ... + A_m$ contains an Bohr neighborhood $B(\Gamma, \delta)$ with*

$$|\Gamma| \ll m^2\alpha^{-\frac{1}{m-2}} \text{ and } \log \delta^{-1} \ll m^3\alpha^{-\frac{1}{m-2}} \log \alpha^{-1}.$$

The dimension of our Bohr set is much smaller than that found by Green and it is this which ensures that the arithmetic progression we find (when there is one at all) is much longer.

Although we do not require it, there is an important strengthening of Lemma 2.2 due to Ruzsa, [Ruz94]. If $M = P_1 + ... + P_d$ where $P_1, ..., P_d$ are arithmetic progressions then we call M a *d-dimensional generalized arithmetic progression*.

Lemma 2.5. *Suppose that $G = \mathbb{Z}/N\mathbb{Z}$. Suppose that $B(\Gamma, \delta)$ is a Bohr set with $|\Gamma| = d$. Then $B(\Gamma, \delta)$ contains a d-dimensional generalized arithmetic progression of size at least $(\delta/d)^d N$.*

Typically this generalized progression occupies a large (roughly $(Cd)^{-d}$) proportion of the Bohr set, and it can be instructive to think of Bohr neighborhoods as generalized progressions. In our results, then, we could use this lemma to draw the stronger conclusion that m-fold sumsets (for $m \geq 3$) contain large multidimensional progressions, but we believe that the results are most easily digested in the form stated.

The paper now splits into five further sections. In the next three we develop the necessary tools for analyzing functions on Bohr sets. The first of these presents the basics, the second establishes our new version of Chang's theorem relative to Bohr sets, and the third recalls some standard density increment lemmas. The last two sections of the paper prove the results we have promised.

3. Local Fourier analysis on compact Abelian groups

Given $f \in L^1(G)$ we often want to approximate f by a less complicated function. One way to do this is to approximate f by its expectation on approximate level sets of characters i.e. sets on which characters do not vary too much. To analyze the error in doing this we restrict the function to these approximate level sets and use the Fourier transform on the restricted function. Specifically, if Γ is a set of characters and $x' + \Gamma^{\perp}$ (a maximal joint level set of the characters in Γ) has positive measure in G then it is easy to localize the Fourier transform to $x' + \Gamma^{\perp}$:

$$L^1(x' + \mu_{\Gamma^{\perp}}) \to \ell^{\infty}(\widehat{G}); f \mapsto \widehat{fd(x' + \mu_{\Gamma^{\perp}})}.$$

Note that the right hand side is constant on cosets of $\Gamma^{\perp\perp}$ and so is really an element of $\ell^{\infty}(\widehat{G}/\Gamma^{\perp\perp})$.

Bourgain, in [Bou99], observed that one can localize the Fourier transform to typical approximate level sets and retain approximate versions of a number of the standard results for the Fourier transform on compact Abelian groups. Since his original work various expositions and extensions of the work have appeared most notably in the various papers of Green and Tao. Indeed all the results of this section can be found in [GT08], for example.

Annihilators are subgroups of G; a Bohr set is a sort of approximate annihilator and, consequently, we would like it to behave like a sort of approximate subgroup. Suppose that $\eta \in (0, 1]$. Then $B(\Gamma, \delta) + B(\Gamma, \eta\delta) \subset B(\Gamma, (1 + \eta)\delta)$. If $B(\Gamma, (1 + \eta)\delta)$ is not much bigger than $B(\Gamma, \delta)$ then we have a sort of approximate additive closure in the sense that $B(\Gamma, \delta) + B(\Gamma, \eta\delta) \approx B(\Gamma, (1 + \eta)\delta)$. Not all Bohr sets have this property. However, Bourgain showed that typically they do. For our purposes we have the following proposition.

Proposition 3.1. *Suppose that G is a compact Abelian group, Γ a set of d characters on G and $\delta \in (0, 1]$. There is an absolute constant $c_{\mathcal{R}} > 0$ and a $\delta' \in [\delta/2, \delta)$ such that*

$$\frac{\mu_G(B(\Gamma, (1 + \kappa)\delta'))}{\mu_G(B(\Gamma, \delta'))} = 1 + O(|\kappa| d) \tag{1}$$

whenever $|\kappa| d \leq c_{\mathcal{R}}$.

This result is not as easy as the rest of the section, it uses a covering argument; a nice proof can be found in [GT08]. We say that δ' is *regular for* Γ or that $B(\Gamma,\delta')$ is *regular* if

$$\frac{\mu_G(B(\Gamma,(1+\kappa)\delta'))}{\mu_G(B(\Gamma,\delta'))}=1+O(|\kappa|d) \text{ whenever } |\kappa|d\leq c_{\mathcal{R}}.$$

It is regular Bohr sets to which we localize the Fourier transform. We require a little more notation regarding measures. As usual if X is a topological space we write $M(X)$ for the regular complex-valued Borel measures on X. If $\mu\in M(G)$ then $\operatorname{supp}\mu$ denotes the support of μ, and if $x\in G$ as well then $x+\mu$ denotes the measure μ translated by x.

We begin by observing that normalized regular Bohr cutoffs are approximately translation invariant and so function as normalized approximate Haar measures.

Lemma 3.2. (Normalized approximate Haar measure) *Suppose that G is a compact Abelian group and $B(\Gamma,\delta)$ is a regular Bohr set. If $y\in B(\Gamma,\delta')$ then $\|(y+\beta_\delta)-\beta_\delta\|\ll d\delta'\delta^{-1}$.*

The proof follows immediately from the definition of regularity. In applications the following two simple corollaries will be useful but they should be ignored until they are used.

Corollary 3.3. *Suppose that G is a compact Abelian group and $B(\Gamma,\delta)$ is a regular Bohr set. If $\mu\in M(B(\Gamma,\delta'))$ then $\|\beta*\mu-\beta\int d\mu\|\ll\|\mu\|d\delta'\delta^{-1}$.*

Corollary 3.4. *Suppose that G is a compact Abelian group and $B(\Gamma,\delta)$ is a regular Bohr set. Suppose that $f\in L^\infty(G)$. If $x-y\in B(\Gamma,\delta')$ then $|f*\beta(x)-f*\beta(y)|\ll\|f\|_\infty d\delta'\delta^{-1}$.*

With an approximate Haar measure we are in a position to define the local Fourier transform: Suppose that Γ is a finite set of characters, δ is regular for Γ and $x'\in G$. Then we define the Fourier transform local to $x'+B(\Gamma,\delta)$ by

$$L^1(x'+\beta_{\Gamma,\delta})\rightarrow\ell^\infty(\widehat{G}); f\mapsto \widehat{fd(x'+\beta_{\Gamma,\delta})},$$

where we take the convention that

$$L^1(\mu):=\{f\in L^1(G):\operatorname{supp}f\subset\operatorname{supp}\mu \text{ and } \int|f|d\mu<\infty\}.$$

The translation of the Bohr set by x' simply twists the Fourier transform and is unimportant for the most part so we tend to restrict ourselves to the case when $x'=0$.

$\widehat{fd\mu_{\Gamma^\perp}}$ was constant on cosets of $\Gamma^{\perp\perp}$. In the approximate setting have an approximate analogue of this. First the analogue of $\Gamma^{\perp\perp}$; there are a number of possibilities:

$$\begin{aligned}\{\gamma:|1-\gamma(x)|\leq\epsilon \text{ for all } x\in B(\Gamma,\delta)\}&\quad\text{for } \epsilon\in(0,1]\\ \{\gamma:|1-\widehat{\beta}(\gamma)|\leq\epsilon\}&\quad\text{for } \epsilon\in(0,1]\\ \{\gamma:|\widehat{\beta}(\gamma)|\geq\epsilon\}&\quad\text{for } \epsilon\in(0,1].\end{aligned}$$

In applications each of these classes of sets is useful and so we should like all of them to be approximately equivalent. There is a clear chain of inclusions between the classes:

$$\{\gamma:|1-\gamma(x)|\leq\epsilon \text{ for all } x\in B(\Gamma,\delta)\}\subset\{\gamma:|1-\widehat{\beta}(\gamma)|\leq\epsilon\}\subset\{\gamma:|\widehat{\beta}(\gamma)|\geq 1-\epsilon\}$$

for $\epsilon \in (0,1]$. For a small cost in the width of the Bohr set we can ensure that the sets in the third class are contained in those in the first.

Lemma 3.5. *Suppose that G is a compact Abelian group and $B(\Gamma,\delta)$ is a regular Bohr set. Suppose that $\eta_1,\eta_2 > 0$. Then there is a $\delta' \gg \eta_1\eta_2\delta/d$ such that*

$$\{\gamma : |\widehat{\beta}(\gamma)| \geq \eta_1\} \subset \{\gamma : |1-\gamma(x)| \leq \eta_2 \textit{ for all } x \in B(\Gamma,\delta')\}.$$

The lemma follows easily from Lemma 3.2.

4. The structure of sets of characters supporting large values of the local Fourier transform

Green and Tao in [GT08] were the first to prove the following proposition which captures a version of Bessel's inequality local to Bohr sets in a form useful for applications.

Proposition 4.1. *Suppose that G is a compact Abelian group and $B(\Gamma,\delta)$ is a regular Bohr set. Suppose that $f \in L^2(\beta)$ and $\epsilon,\eta \in (0,1]$. Then there is a set of characters Λ and a $\delta' \in (0,1]$ such that*

$$|\Lambda| \ll \epsilon^{-2}\|f\|_{L^1(\beta)}^{-2}\|f\|_{L^2(\beta)}^2 \textit{ and } \delta' \gg \eta\delta/d,$$

and

$$\{\gamma \in \widehat{G} : |\widehat{fd\beta}(\gamma)| \geq \epsilon\|f\|_{L^1(\beta)}\} \subset \{\gamma \in \widehat{G} : |1-\gamma(x)| \leq \eta \textit{ for all } x \in B(\Gamma\cup\Lambda,\delta')\}.$$

It is instructive to consider this result in the case when $B(\Gamma,\delta) = G$. We are then given a set of characters Λ and a $\delta' \in (0,1]$ such that

$$|\Lambda| \ll \epsilon^{-2}\|f\|_{L^1(\mu_G)}^{-2}\|f\|_{L^2(\mu_G)}^2 \text{ and } \delta' \gg \eta,$$

and

$$\{\gamma \in \widehat{G} : |\widehat{f}(\gamma)| \geq \epsilon\|f\|_{L^1(\mu_G)}\} \subset \{\gamma \in \widehat{G} : |1-\gamma(x)| \leq \eta \text{ for all } x \in B(\Lambda,\delta')\}.$$

Now it is natural to ask what this has to do with Bessel's inequality. Write $\Gamma := \{\gamma \in \widehat{G} : |\widehat{f}(\gamma)| \geq \epsilon\|f\|_{L^1(\mu_G)}\}$. Then

$$|\Gamma|.\epsilon^2\|f\|_{L^1(\mu_G)}^2 \leq \sum_{\gamma\in\Gamma}|\widehat{f}(\gamma)|^2 \leq \sum_{\gamma\in\widehat{G}}|\widehat{f}(\gamma)|^2 \leq \|f\|_{L^2(\mu_G)}^2$$

by Bessel's inequality. It follows that

$$|\Gamma| \leq \epsilon^{-2}\|f\|_{L^1(\mu_G)}^{-2}\|f\|_{L^2(\mu_G)}^2,$$

and, moreover, it is easy to check that there is a $\delta' \gg \eta$ such that

$$\Gamma \subset \{\gamma \in \widehat{G} : |1-\gamma(x)| \leq \eta \text{ for all } x \in B(\Gamma,\delta')\}.$$

Setting $\Lambda := \Gamma$ yields the conclusion of Proposition 4.1 in this special case. Restricting functions to Bohr sets complicates matters. However, there are some easy rules of thumb to bear in mind. The bound on $|\Lambda|$ is very important. The dependence of δ' on δ needs to be linear because we intend to iterate the procedure, however, the ratio $\delta^{-1}\delta'$ can be very much smaller without any tangible cost.

In the next proposition we trade a worse ratio $\delta^{-1}\delta'$, which has little impact, for a large improvement in the bound on $|\Lambda|$.

Proposition 4.2. *Suppose that G is a compact Abelian group and $B(\Gamma,\delta)$ is a regular Bohr set. Suppose that $f \in L^2(\beta)$ and $\epsilon,\eta \in (0,1]$. Then there is a set of characters Λ and a $\delta' \in (0,1]$ such that*

$$|\Lambda| \ll \epsilon^{-2}\log\|f\|_{L^1(\beta)}^{-2}\|f\|_{L^2(\beta)}^2 \text{ and } \delta' \gg \delta\eta\epsilon^2/d^2\log\|f\|_{L^1(\beta)}^{-2}\|f\|_{L^2(\beta)}^2,$$

and

$$\{\gamma \in \widehat{G} : |\widehat{fd\beta}(\gamma)| \geq \epsilon\|f\|_{L^1(\beta)}\} \subset \{\gamma \in \widehat{G} : |1-\gamma(x)| \leq \eta \text{ for all } x \in B(\Gamma\cup\Lambda,\delta')\}.$$

This result can be seen as a local version of Chang's theorem (from [Cha02]); indeed, the proof is essentially a combination of the ideas used to prove that theorem and those used to prove Proposition 4.1. The key tool in Chang's theorem is that of dissociativity; in the local setting we use the following version of it. If Λ is a set of characters on G and $m : \Lambda \to \mathbb{Z}$ has finite support then put

$$m.\Lambda := \sum_{\lambda\in\Lambda} m_\lambda.\lambda \text{ and } |m| := \sum_{\lambda\in\Lambda}|m_\lambda|,$$

where the second '.' in the first definition denotes the natural action of $\mathbb{Z}$ on $\widehat{G}$. If S is a non-empty symmetric neighborhood of $0_{\widehat{G}}$ then we say that Λ is *S-dissociated* if

$$m.\Lambda \in S \Rightarrow m \equiv 0.$$

The usual definition of dissociativity corresponds to taking $S = \{0_{\widehat{G}}\}$.

Proposition 4.2 follows straightforwardly from the next two lemmas.

Lemma 4.3. *Suppose that G is a compact Abelian group and $B(\Gamma,\delta)$ is a regular Bohr set. Suppose that $\eta',\eta \in (0,1]$ and Δ is a set of characters on G. If Λ is a maximal $\{\gamma : |\widehat{\beta}(\gamma)| \geq \eta'\}$-dissociated subset of Δ then there is a $\delta' \gg \min\{\eta/|\Lambda|,\eta'\eta\delta/d\}$ such that*

$$\Delta \subset \{\gamma : |1-\gamma(x)| \leq \eta \text{ for all } x \in B(\Gamma\cup\Lambda,\delta')\}.$$

Lemma 4.4. *Suppose that G is a compact Abelian group and $B(\Gamma,\delta)$ is a regular Bohr set. Suppose that $0 \not\equiv f \in L^2(\beta)$, $k \in \mathbb{N}$ and $\epsilon,\eta \in (0,1]$. Then there is a $\delta' \gg \delta/dk$ regular for Γ such that if Λ is a $\{\gamma : |\widehat{\beta'}(\gamma)| \geq 1/3\}$-dissociated subset of $\{\gamma \in \widehat{G} : |\widehat{fd\beta}(\gamma)| \geq \epsilon\|f\|_{L^1(\beta)}\}$ with size at most k, then*

$$|\Lambda| \ll \epsilon^{-2}\log\|f\|_{L^1(\beta)}^{-2}\|f\|_{L^2(\beta)}^2.$$

4.5. **Proof of Lemma 4.3.** The lemma is really rather simple to prove although technical. It rests on localizing the following simple observation of Chang [Cha02].

Lemma 4.6. *Suppose that G is a compact Abelian group. Suppose that Δ is a set of characters on G and Λ is a maximal dissociated subset of Δ. Then $\Delta \subset \langle\Lambda\rangle$.*

Here $\langle\Lambda\rangle$ denotes all the finite $\pm$-sums of elements of Λ i.e.

$$\langle\Lambda\rangle := \{m.\Lambda : m : \Lambda \to \{-1,0,1\} \text{ and } |\operatorname{supp} m| < \infty\}.$$

For our purposes we have the following.

Lemma 4.7. *Suppose that G is a compact Abelian group. Suppose that S is a non-empty symmetric neighborhood of $0_{\widehat{G}}$. Suppose that Δ is a set of characters on G and Λ is a maximal S-dissociated subset of Δ. Then $\Delta \subset \langle\Lambda\rangle + S$.*

Proof. If $\lambda_0 \in \Delta \setminus (\langle\Lambda\rangle + S)$ then we put $\Lambda' := \Lambda \cup \{\lambda_0\}$, which is a strict superset of Λ, and a subset of Δ. It turns out that Λ' is also S-dissociated which contradicts the maximality of Λ. Suppose that $m : \Lambda' \to \{-1, 0, 1\}$ and $m.\Lambda' \in S$. Then we have three possibilities for the value of m_{λ_0}:

(1) $m.\Lambda' = \lambda_0 + m|_\Lambda.\Lambda$, in which case $\lambda_0 \in -m|_\Lambda.\Lambda + S \subset \langle\Lambda\rangle + S$ - a contradiction;
(2) $m.\Lambda' = -\lambda_0 + m|_\Lambda.\Lambda$, in which case $\lambda_0 \in m|_\Lambda.\Lambda - S \subset \langle\Lambda\rangle + S$ - a contradiction;
(3) $m.\Lambda' = m|_\Lambda.\Lambda$, in which case $m|_\Lambda \equiv 0$ since Λ is S-dissociated and hence $m \equiv 0$.

It follows that $m.\Lambda' \in S \Rightarrow m \equiv 0$ i.e. Λ' is S-dissociated as claimed. □

Lemma 4.3 then follows from the above and the following lemma.

Lemma 4.8. *Suppose that G is a compact Abelian group and $B(\Gamma, \delta)$ is a regular Bohr set. Suppose that $\eta', \eta \in (0, 1]$ and Λ is a set of characters on G. Then there is a $\delta' \gg \min\{\eta/|\Lambda|, \eta'\eta\delta/d\}$ such that*

$$\langle\Lambda\rangle + \{\gamma : |\widehat{\beta}(\gamma)| \geq \eta'\} \subset \{\gamma : |1 - \gamma(x)| \leq \eta \textit{ for all } x \in B(\Gamma \cup \Lambda, \delta')\}.$$

Proof. The lemma has two parts.

(1) If $\lambda \in \langle\Lambda\rangle$ then
$$|1 - \lambda(x)| \leq \sum_{\lambda' \in \Lambda} |1 - \lambda'(x)|,$$
so there is a $\delta'' \gg \eta/|\Lambda|$ such that
$$\langle\Lambda\rangle \subset \{\gamma : |1 - \gamma(x)| \leq \eta/2 \text{ for all } x \in B(\Lambda, \delta'')\}.$$
(2) By Lemma 3.5 there is a $\delta''' \gg \eta\eta'\delta/d$ such that
$$\{\gamma : |\widehat{\beta}(\gamma)| \geq \eta'\} \subset \{\gamma : |1 - \gamma(x)| \leq \eta/2 \text{ for all } x \in B(\Gamma, \delta''')\}.$$

Taking $\delta' = \min\{\delta'', \delta'''\}$ we have the result by the triangle inequality. □

4.9. **Proof of Lemma 4.4.** The proof has three main ingredients.

- *(Rudin's inequality)* In [Cha02] Chang uses the following dual statement of Rudin's inequality to prove her theorem.

  Proposition 4.10. *Suppose that G is a compact Abelian group. If Λ is dissociated then*
  $$\|\widehat{f}|_\Lambda\|_2 \ll \sqrt{p}\|f\|_{p'} \textit{ for all } f \in L^{p'}(G)$$
  and all conjugate exponents p and p' with $p' \in (1, 2]$.

  For a proof of this see, for example, Chapter 5 of Rudin [Rud90].
- *(Almost-orthogonality lemma)* To prove Proposition 4.1, Green and Tao localized Bessel's inequality to Bohr sets by using the following almost-orthogonality lemma.

  Lemma 4.11. (Cotlar's almost orthogonality lemma) *Suppose that v and (w_j) are elements of a complex inner product space. Then*
  $$\sum_j |\langle v, w_j\rangle|^2 \leq \langle v, v\rangle \max_j \sum_i |\langle w_i, w_j\rangle|.$$

- *(Smoothed measures)* Suppose that $B(\Gamma,\delta)$ is a regular Bohr set. We produce a range of smoothed alternatives to the measure β; specifically suppose that $L\in\mathbb{N}$ and $\kappa\in(0,1]$. Then we may define
$$\tilde{\beta}^{L,\kappa}_{\Gamma,\delta} := \beta_{\Gamma,(1+\kappa)\delta} * \beta^{L}_{\Gamma,\kappa\delta/L},$$
where $\beta^{L}_{\Gamma,\kappa\delta/L}$ denotes the convolution of $\beta_{\Gamma,\kappa\delta/L}$ with itself L times. This measure has the property that it is supported on $B(\Gamma,(1+2\kappa)\delta)$ and uniform on $B(\Gamma,\delta)$, indeed
$$\tilde{\beta}^{L,\kappa}_{\Gamma,\delta}|_{B(\Gamma,\delta)} = \frac{\mu_G|_{B(\Gamma,\delta)}}{\mu_G(B(\Gamma,(1+\kappa)\delta))} = \frac{\mu_G(B(\Gamma,\delta))}{\mu_G(B(\Gamma,(1+\kappa)\delta))}.\beta_{\Gamma,\delta}. \tag{2}$$
It follows that every $f\in L^1(\beta_{\Gamma,\delta})$ has $\widehat{fd\beta_{\Gamma,\delta}}$ well approximated by $\widehat{fd\tilde{\beta}^{L,\kappa}_{\Gamma,\delta}}$. Specifically
$$\widehat{fd\tilde{\beta}^{L,\kappa}_{\Gamma,\delta}}(\gamma) = (1+O(\kappa d))\widehat{fd\beta_{\Gamma,\delta}}(\gamma) \tag{3}$$
by regularity of $B(\Gamma,\delta)$.

We use almost-orthogonality and the smoothed measures to show the following localization of Rudin's inequality. The proof of the lemma to which this section is devoted then follows the usual proof of Chang's theorem.

Lemma 4.12. *Suppose that G is a compact Abelian group and $B(\Gamma,\delta)$ is a regular Bohr set. Suppose that Λ is a set of characters. Then there is a $\delta'\gg\delta/d|\Lambda|$ regular for Γ such that if Λ is $\{\gamma : |\widehat{\beta'}(\gamma)|\geq 1/3\}$-dissociated then*
$$\|\widehat{fd\beta}|_{\Lambda}\|_2 \ll \sqrt{p}\|f\|_{L^{p'}(\beta)} \text{ for all } f\in L^{p'}(\beta)$$
and all conjugate exponents p and p' with $p'\in(1,2]$.

Proof. Begin by fixing the level of smoothing (i.e. the parameters κ and L of $\tilde{\beta}^{L,\kappa}_{\Gamma,\delta}$) that we require and write $\tilde{\beta}$ for $\tilde{\beta}^{L,\kappa}_{\Gamma,\delta}$. Set $L:=2k$ and recall (3):
$$\widehat{gd\tilde{\beta}}(\gamma) = (1+O(\kappa d))\widehat{gd\beta}(\gamma) \text{ for all } g\in L^1(\beta);$$
so we can pick $\kappa'\gg d^{-1}$ such that for all $\kappa\leq\kappa'$
$$\frac{1}{2}|\widehat{gd\beta}(\gamma)| \leq |\widehat{gd\tilde{\beta}}(\gamma)| \leq \frac{3}{2}|\widehat{gd\beta}(\gamma)| \text{ for all } g\in L^1(\beta).$$
By Proposition 3.1, we can take κ with $\kappa'\geq\kappa\gg d^{-1}$ such that $\delta':=\kappa\delta/L$ is regular.

Define the Riesz product
$$q(x) := \prod_{\lambda\in\Lambda}\left(1+\frac{\lambda(x)+\overline{\lambda}(x)}{2}\right).$$
Every term in the product is non-negative and so q is non-negative and it is fairly easy to compute the Fourier transform of q:
$$\widehat{q}(\gamma) = \sum_{m:m.\Lambda=\gamma} 2^{-|m|}. \tag{4}$$
Since Λ is $\{\gamma : |\widehat{\beta'}(\gamma)|\geq 1/3\}$-dissociated, it is certainly vanilla dissociated and hence $\widehat{q}(0_{\widehat{G}})=1$ and so, by non-negativity of q, $\|q\|_1=1$.

Use q to define the map

$$T : L^1(\beta) \rightarrow L^1(G); g \mapsto (gd\beta) * q,$$

and note that

$$\|Tg\|_1 = \|(gd\beta) * q\|_1 \leq \|g\|_{L^1(\beta)}\|q\|_1 = \|g\|_{L^1(\beta)}$$

by the triangle inequality. We now claim a corresponding result for $\|Tg\|_2$, the proof of which we defer until we have finished the proof of the lemma.

Claim 1. *If $g \in L^2(\beta)$ then $\|Tg\|_2 \ll \|g\|_{L^2(\beta)}$.*

Assuming this claim, by the Riesz-Thorin interpolation theorem we have

$$\|Tg\|_{p'} \ll \|g\|_{L^{p'}(\beta)} \text{ for any } p' \in [1, 2]. \tag{5}$$

Hence, if $f \in L^{p'}(\beta)$,

$$\begin{aligned}\frac{1}{2}\|\widehat{fd\beta}|_\Lambda\|_2 &\leq \|\widehat{fd\beta}\widehat{q}|_\Lambda\|_2 \text{ since } \widehat{q}(\lambda) \geq 1/2 \text{ for all } \lambda \in \Lambda,\\ &= \|\widehat{Tf}|_\Lambda\|_2 \text{ by the definition of } T,\\ &\ll \sqrt{p}\|Tf\|_{p'} \text{ by Rudin's inequality,}\\ &\ll \sqrt{p}\|f\|_{L^{p'}(\beta)} \text{ by (5).}\end{aligned}$$

The lemma follows. It remains to prove the claim.

Proof of Claim. Begin by noting the following consequence of (2).

$$\|Tg\|_2^2 = \left(\frac{\mu_G(B(\Gamma, \delta(1+\kappa)))}{\mu_G(B(\Gamma, \delta))}\right)^2 \|(gd\tilde{\beta}) * q\|_2^2. \tag{6}$$

By Plancherel's theorem

$$\|(gd\tilde{\beta}) * q\|^2 = \sum_{\gamma \in \widehat{G}} |\widehat{(gd\tilde{\beta})}(\gamma)\widehat{q}(\gamma)|^2 = \sum_{\gamma \in \widehat{G}} |\langle g, \widehat{q}(\gamma)\gamma\rangle_{L^2(\tilde{\beta})}|^2.$$

Cotlar's Almost Orthoginality Lemma applied to the second sum gives

$$\begin{aligned}\|(gd\tilde{\beta}) * q\|^2 &\leq \langle g, g\rangle_{L^2(d\tilde{\beta})} \max_\gamma \sum_{\gamma'} |\langle \widehat{q}(\gamma)\gamma, \widehat{q}(\gamma')\gamma'\rangle_{L^2(\tilde{\beta})}|\\ &\leq \|g\|^2_{L^2(d\tilde{\beta})} \max_\gamma \sum_{\gamma'} \widehat{q}(\gamma')|\widehat{\tilde{\beta}}(\gamma - \gamma')|.\end{aligned}$$

For any $\gamma \in \widehat{G}$ we can estimate the last sum in a manner independent of γ by using a positivity argument:

$$\begin{aligned}
\sum_{\gamma' \in \widehat{G}} \widehat{q}(\gamma')|\widehat{\tilde{\beta}}(\gamma - \gamma')| &= \sum_{\gamma' \in \widehat{G}} \widehat{q}(\gamma - \gamma')|\widehat{\beta}(\gamma')\widehat{\beta'}(\gamma')^L| \text{ by definition of } \tilde{\beta},\\
&\leq \sum_{\gamma' \in \widehat{G}} \widehat{q}(\gamma - \gamma')|\widehat{\beta'}(\gamma')|^L \text{ since } |\widehat{\beta}(\gamma')| \leq \|\beta\| = 1 \text{ and } \widehat{q} \geq 0,\\
&= \widehat{qd\beta'^L}(\gamma) \text{ since } L \text{ is even and } \widehat{q} \geq 0,\\
&\leq \|q\|_{L^1(\beta'^L)}\\
&= \widehat{qd\beta'^L}(0_{\widehat{G}}) \text{ by non-negativity of } qd\beta'^L,\\
&= \sum_{\gamma' \in \widehat{G}} \widehat{q}(\gamma')|\widehat{\beta'}(\gamma')|^L \text{ by symmetry of } \widehat{q}.
\end{aligned}$$

We estimate this in turn by splitting the range of summation into two parts:

$$\sum_{\gamma' \in \widehat{G}} \widehat{q}(\gamma')|\widehat{\beta'}(\gamma')|^L \leq \sum_{\gamma' : |\widehat{\beta'}(\gamma')| \geq 1/3} \widehat{q}(\gamma')|\widehat{\beta'}(\gamma')|^L + \sum_{\gamma' : |\widehat{\beta'}(\lambda)| \leq 1/3} \widehat{q}(\gamma')|\widehat{\beta'}(\gamma')|^L. \tag{7}$$

(1) For the first sum: $|\widehat{q}(\gamma')| \leq \|q\|_1 = 1$ and $|\widehat{\beta'}(\gamma')^L| \leq \|\beta'^L\| = 1$ so that each summand is at most 1, furthermore $\operatorname{supp} \widehat{q} \subset \langle \Lambda \rangle$ so

$$\sum_{\gamma' : |\widehat{\beta'}(\gamma')| \geq 1/3} \widehat{q}(\gamma')|\widehat{\beta'}(\gamma')|^L \leq \sum_{\gamma' \in \langle \Lambda \rangle : |\widehat{\beta'}(\gamma')| \geq 1/3} 1.$$

This range of summation contains at most 1 element by $\{\gamma : |\widehat{\beta'}(\gamma)| \geq 1/3\}$-dissociativity of Λ, and hence the sum is bounded above by 1.

(2) For the second sum: $|\widehat{q}(\gamma')| \leq \|q\|_1 = 1$ and $|\widehat{\beta'}(\gamma')^L| \leq 3^{-L}$ for γ' in the range of summation so that each summand is at most $9^{-|\Lambda|}$, however $\operatorname{supp} \widehat{q} \subset \langle \Lambda \rangle$ and $|\langle \Lambda \rangle| \leq 3^{|\Lambda|}$ so

$$\sum_{\gamma' : |\widehat{\beta'}(\gamma')| \leq 1/3} \widehat{q}(\gamma')|\widehat{\beta'}(\gamma')|^L \leq \sum_{\gamma' \in \langle \Lambda \rangle} 9^{-|\Lambda|} \leq 1.$$

It follows that the right hand side of (7) is bounded above by 2 and hence that

$$\|(gd\tilde{\beta}) * q\|^2 \leq 2\|g\|^2_{L^2(d\tilde{\beta})}.$$

This, (6) and (2) yield

$$\|Tg\|_2^2 \leq 2\frac{\mu_G(B(\Gamma, \delta(1+\kappa)))}{\mu_G(B(\Gamma, \delta))}\|g\|^2_{L^2(\beta)},$$

from which the result follows by regularity. □

□

Proof of Lemma 4.4. By the localized dual of Rudin's inequality (Lemma 4.12) for any $p' \in (1, 2]$ we have

$$|\Lambda|.\epsilon^2\|f\|^2_{L^1(\beta)} \leq \sum_{\lambda \in \Lambda} |\widehat{fd\beta}(\lambda)|^2 = \|\widehat{fd\beta}|_\Lambda\|_2^2 \ll p\|f\|^2_{L^{p'}(\beta)},$$

where p is the conjugate exponent of p'. The log-convexity of $\|.\|_{L^{p'}(\beta)}$ gives

$$|\Lambda| \ll \epsilon^{-2} p \left(\frac{\|f\|_{L^2(\beta)}}{\|f\|_{L^1(\beta)}}\right)^{\frac{4}{p}}.$$

Optimizing p gives the result. □

5. Local Fourier analysis and the iteration method

The tools of local Fourier analysis were originally developed with iteration in mind. Specifically if A is a subset of a regular Bohr set B then we shall often have an argument which tells us that there is a large ℓ^p-mass of the local Fourier transform $\widehat{\chi_A d\beta}$ and, as is the case in the non-local setting, this leads to a density increment on a (sub-)Bohr neighborhood. For our purposes we have the following two standard lemmas which take a large ℓ^∞ and ℓ^2 Fourier-space mass and convert it into a density increment.

Lemma 5.1. (ℓ^∞ density increment argument) *Suppose that G is a compact Abelian group and $B = B(\Gamma, \delta)$ is a regular Bohr set. Suppose that $A \subset B$ has relative density α and write $f := \chi_A - \alpha\chi_B$. Suppose that*

$$|\widehat{fd\beta}(\gamma)| \geq \eta\alpha \text{ for some } \gamma \in \widehat{G}.$$

Then there is a regular Bohr set $B' := B(\Gamma', \delta')$ with $\Gamma' := \Gamma \cup \{\gamma\}$ and $\delta \geq \delta' \gg \eta\alpha\delta/d$ such that

$$\|\chi_A * \beta'\|_{L^\infty(\beta)} \geq \alpha\left(1 + 2^{-3}\eta\right).$$

Proof. Let $\delta' \in (0, 1]$ be a constant to be determined later. A trivial instance of Hausdorff's inequality tells us that

$$\|(fd\beta) * \beta'\| \geq |\widehat{fd\beta}(\gamma)||\widehat{\beta'}(\gamma)| \geq \eta\alpha|\widehat{\beta'}(\gamma)|. \tag{8}$$

Since $B' \subset B(\{\gamma\}, \delta')$ we have $|\widehat{\beta'}(\gamma)| \geq 1 - O(\delta')$. It follows that there is a $\delta_0' \gg 1$ such that if $\delta' \leq \delta_0'$ then $|\widehat{\beta'}(\gamma)| \geq 1/2$. Now

$$\int d((fd\beta) * \beta') = 0,$$

hence by (8)

$$\int d((fd\beta) * \beta')_+ \geq \eta\alpha|\widehat{\beta'}(\gamma)|/2 \geq \eta\alpha/4.$$

If follows from the regularity of B and the fact that $B' \subset B(\Gamma, \delta')$ that

$$\|(fd\beta) * \beta' - (f * \beta')d\beta\| = O(d\delta'\delta^{-1}),$$

and so

$$\int (f * \beta')_+ d\beta \geq \eta\alpha/4 + O(d\delta'\delta^{-1}).$$

By regularity of $B(\Gamma, \delta)$ we have

$$\int (f * \beta')_+ d\beta \leq \|\chi_A * \beta'\|_\infty - \alpha + O(d\delta'\delta^{-1}),$$

so

$$\|\chi_A * \beta'\|_\infty \geq \alpha(1 + 1/4) + O(d\delta'\delta^{-1}).$$

Hence by Proposition 3.1 we can pick $\delta' \gg \eta\alpha\delta/d$ regular for Γ' with $\delta' \leq \delta_0'$ and $\delta' \leq \delta$, and such that the conclusion of the lemma holds. □

Lemma 5.2. (ℓ^2 density increment argument) *Suppose that G is a compact Abelian group and $B = B(\Gamma, \delta)$ is a regular Bohr set. Suppose that $A \subset B$ has relative density α and write $f := \chi_A - \alpha\chi_B$. Suppose that $B' = B(\Gamma', \delta')$ is a Bohr set with $\Gamma \subset \Gamma'$ and*

$$\langle f * \beta', (f d\beta) * \beta' \rangle \geq c\alpha^2.$$

Then

$$\|\chi_A * \beta'\|_\infty \geq \alpha(1 + c) + O(d\delta'\delta^{-1}).$$

Proof. We expand the inner product:

$$\begin{aligned}\langle f * \beta', (f d\beta) * \beta' \rangle = \langle \chi_A * \beta', (\chi_A d\beta) * \beta' \rangle - \alpha\langle \chi_B * \beta', (\chi_A d\beta) * \beta' \rangle \\ -\alpha\langle \chi_A * \beta', \beta * \beta' \rangle + \alpha^2\langle \chi_B * \beta', \beta * \beta' \rangle.\end{aligned}$$

Now we estimate each term. First

$$\begin{aligned}\langle \chi_A * \beta', (\chi_A d\beta) * \beta' \rangle &\leq \|\chi_A * \beta'\|_\infty \|(\chi_A d\beta) * \beta'\| \\ &\leq \|\chi_A * \beta'\|_\infty \|\chi_A\|_{L^1(\beta)} \|\beta'\| = \|\chi_A * \beta'\|_\infty \alpha.\end{aligned}$$

By Lemma 3.2 and the fact that $B(\Gamma', \delta') \subset B(\Gamma, \delta')$ we have

$$\|\beta * \beta' * \beta' - \beta\| = O(d\delta'\delta^{-1})$$

whence

$$\langle \chi_B * \beta', (\chi_A d\beta) * \beta' \rangle = \langle \beta * \beta' * \beta', \chi_A \rangle = \alpha + O(d\delta'\delta^{-1}),$$
$$\langle \chi_A * \beta', \beta * \beta' \rangle = \langle \chi_A, \beta * \beta' * \beta' \rangle = \alpha + O(d\delta'\delta^{-1}),$$

and

$$\langle \chi_B * \beta', \beta * \beta' \rangle = \langle \chi_B, \beta * \beta' * \beta' \rangle = 1 + O(d\delta'\delta^{-1}).$$

It follows that

$$\alpha\|\chi_A * \beta'\|_\infty \geq \alpha^2(1 + c) + O(\alpha d\delta'\delta^{-1}),$$

from which we get the result on division by α. □

6. Proof of Theorem 1.4

We begin with a brief overview of the argument in the model setting. This argument can be made to prove the following result, which was first established by Green in [Gre02b].

Theorem 6.1. *Suppose that G is a finite dimensional compact vector space over $\mathbb{F}_2$ and $A \subset G$ has density $\alpha > 0$. Then $A + A$ contains a subspace of dimension $2^{-4}\alpha^2 \dim G$.*

There are three main ingredients to the proof of this result. First we have the iteration lemma - the driving force. In words it says that either $A + A$ contains most of G or we can find an affine subspace on which A has increased density.

Lemma 6.2. (Model iteration lemma) *Suppose that G is a compact vector space over $\mathbb{F}_2$. Suppose that $A \subset G$ has density α. Suppose that $\sigma \in (0, 1]$. Then at least one of the following is true.*

1. *(The sumset contains most of G) $A + A$ contains at least a proportion $1 - \sigma$ of G.*
2. *(Density increment) There is a subspace V of G such that*

$$\|\chi_A * \mu_V\|_\infty \geq \alpha(1 + 1/4) \textit{ and } \operatorname{cod} V \leq 8\alpha^{-2} \log \sigma^{-1}.$$

The proof of this is not difficult; we sketch the main ideas now. It was a crucial insight of Green in [Gre02a] to get control of $A \subset G$ by looking at its complement. Specifically if $S \subset (A+A)^c$ then we have

$$\langle \chi_A * \chi_A, \chi_S \rangle = 0.$$

Green employed an ingenious argument to exploit this information; ours is less sophisticated. Plancherel's theorem and the triangle inequality in the usual fashion will give

$$\sum_{\gamma \neq 0_{\widehat{G}}} |\widehat{\chi_A}(\gamma)|^2 |\widehat{\chi_S}(\gamma)| \geq \alpha^2 \sigma,$$

so if $\mathcal{L}$ is the set of non-trivial characters supporting large values of $|\widehat{\chi_S}|$ then it follows easily enough that

$$\sum_{\gamma \in \mathcal{L}} |\widehat{\chi_A}(\gamma)|^2 \gg \alpha^2.$$

Such a bound provides an ℓ^2 density increment for A; we bound the codimension of the subspace on which we get the increment by using Chang's theorem.

The second ingredient is a simple pigeonhole argument which says that if a set contains a large proportion of a vector space then it must contain a large affine subspace.

Lemma 6.3. (Pigeonhole lemma) *Suppose that G is a finite dimensional compact vector space over $\mathbb{F}_2$ and that $A \subset G$ has density $\alpha > 1 - \sigma$. Then A contains a coset of a subspace of dimension $\lfloor \log_2 \sigma^{-1} \rfloor$ provided G contains a subspace of dimension $\lfloor \log_2 \sigma^{-1} \rfloor$.*

The iteration necessary to prove Theorem 6.1 is now very simple. At each stage of the argument we apply the iteration lemma and conclude that either $A+A$ contains a large portion of an affine space or the density of A can be increased on an affine subspace. The density of A cannot be increased forever, so eventually $A+A$ contains a large portion of an affine space and so we may apply the pigeonhole lemma to conclude that $A+A$ contains a large affine space. Optimizing the parameter σ gives the result.

We now turn to the matter of transferring these ideas to the general setting.

Lemma 6.4. (Iteration lemma) *Suppose that G is a compact Abelian group and $B(\Gamma, \delta)$ is a regular Bohr set. Suppose that $A_1, A_2 \subset B(\Gamma, \delta)$. Write α for the geometric mean of the densities of A_1 and A_2 in $B(\Gamma, \delta)$. Suppose that $\sigma \in (0, 1]$. Then at least one of the following is true.*

1. *(The sumset contains most of a Bohr set) There is a regular Bohr set $B(\Gamma, \delta')$ such that $A_1 + A_2$ contains at least a proportion $1 - \sigma$ of $B(\Gamma, \delta')$ and $\delta' \gg \alpha^4 \delta / d$.*
2. *(Density increment) There is a regular Bohr set $B(\Gamma \cup \Lambda, \delta'')$ such that*

$$\|\chi_{A_1} * \beta_{\Gamma \cup \Lambda, \delta''}\|_\infty \|\chi_{A_2} * \beta_{\Gamma \cup \Lambda, \delta''}\|_\infty \geq \alpha^2 (1 + 2^{-4}),$$

and

$$|\Lambda| \ll \alpha^{-2} \log \sigma^{-1} \text{ and } \delta'' \gg \delta \alpha^6 / |\Gamma|^3 \log \sigma^{-1}.$$

As is typical of arguments of this type the proof is quite technical; to simplify the presentation we extract two lemmas from the main argument and place them at the end. Pedagogically, it would be most appropriate to present them now, but

they are hard to motivate without following main proof; hence the order we have chosen.

Proof. We write α_1 and α_2 for the densities of A_1 and A_2 (respectively) in $B(\Gamma,\delta)$, and d for the size of Γ. We may assume that $\alpha_1,\alpha_2>0$ since otherwise the result is trivial.

Let $\delta'\in(0,1]$ be a constant, the value of which will fall out of the proof and write B' for the Bohr set $B(\Gamma,\delta')$ and B for the Bohr set $B(\Gamma,\delta)$. Either we are in the first case of the lemma or we may pick $S\subset B'\setminus(A_1+A_2)$ with $\beta'(S)=\sigma$. We have

$$\langle\chi_{A_1}*(\chi_{A_2}d\beta),\chi_S\rangle_{L^2(\beta')}=0. \tag{9}$$

Write f_i for the *balanced function* $\chi_{A_i}-\alpha_i\chi_B$ of A_i in B. Then

$$f_1*(f_2d\beta)=\chi_{A_1}*(\chi_{A_2}d\beta)-\alpha_1\chi_B*(\chi_{A_2}d\beta)-\chi_{A_1}*\alpha_2\beta+\alpha_1\alpha_2\chi_B*\beta.$$

For $x\in B'$, the last three terms on the right may be estimated using Corollary 3.4:

$$\alpha_1\chi_B*(\chi_{A_2}d\beta)(x)=\alpha_1\chi_{A_2}*\beta(x)=\alpha_1(\alpha_2+O(d\delta'\delta^{-1}));$$

$$\chi_{A_1}*\alpha_2\beta(x)=\alpha_2(\alpha_1+O(d\delta'\delta^{-1}));$$

$$\alpha_1\alpha_2\chi_B*\beta(x)=\alpha_1\alpha_2(1+O(d\delta'\delta^{-1})).$$

Whence

$$f_1*(f_2d\beta)=\chi_{A_1}*(\chi_{A_2}d\beta)-\alpha^2+O(d\delta'\delta^{-1}).$$

It follows from this and (9) that

$$\langle f_1*(f_2d\beta),\chi_S\rangle_{L^2(\beta')}=-\alpha^2\sigma+O(d\delta'\delta^{-1}\sigma).$$

Apply Plancherel's theorem to this inner product to produce a Fourier statement:

$$\sum_{\gamma\in\widehat{G}}\widehat{f_1}(\gamma)\widehat{f_2d\beta}(\gamma)\overline{\widehat{\chi_Sd\beta'}(\gamma)}=-\alpha^2\sigma+O(d\delta'\delta^{-1}\sigma),$$

so by the triangle inequality

$$\sum_{\gamma\in\widehat{G}}|\widehat{f_1}(\gamma)||\widehat{f_2d\beta}(\gamma)||\widehat{\chi_Sd\beta'}(\gamma)|\geq\alpha^2\sigma+O(d\delta'\delta^{-1}\sigma). \tag{10}$$

Let $\mathcal{L}$ be the set of characters supporting the large values of $|\widehat{\chi_Sd\beta'}|$:

$$\mathcal{L}:=\{\gamma\in\widehat{G}:|\widehat{\chi_Sd\beta'}(\gamma)|\geq\alpha\sigma/2\}.$$

The characters supporting small values of $|\widehat{\chi_Sd\beta'}|$ only support a small amount of the sum in (10); specifically by Lemma 6.5 applied with $h=|\widehat{\chi_Sd\beta'}(\gamma)|\chi_{\mathcal{L}^c}(\gamma)$ we have

$$\sum_{\gamma\notin\mathcal{L}}|\widehat{f_1}(\gamma)||\widehat{f_2d\beta}(\gamma)||\widehat{\chi_Sd\beta'}(\gamma)|<\frac{\alpha^2\sigma}{2}.$$

Inserting this into (10) we conclude that

$$\sum_{\gamma\in\mathcal{L}}|\widehat{f_1}(\gamma)||\widehat{f_2d\beta}(\gamma)||\widehat{\chi_Sd\beta'}(\gamma)|\geq\frac{\alpha^2\sigma}{2}+O(d\delta'\delta^{-1}\sigma).$$

We have the trivial inequality $|\widehat{\chi_S d\beta'}(\gamma)| \leq \sigma$ and so (since $\sigma > 0$) we divide through by σ to get

$$\sum_{\gamma \in \mathcal{L}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)| \geq \frac{\alpha^2}{2} + O(d\delta'\delta^{-1}).$$

By Proposition 4.2 there is a set of characters Λ and a $\delta_0'' \in (0,1]$ with

$$|\Lambda| \ll \alpha^{-2}\log\sigma^{-1} \text{ and } \delta_0'' \gg \frac{\delta'\alpha^2}{d^2\log\sigma^{-1}}$$

such that

$$\mathcal{L} \subset \{\gamma : |1-\gamma(x)| \leq 1/2 \text{ for all } x \in B(\Gamma\cup\Lambda, \delta_0'')\}.$$

Suppose that $\delta'' \leq \delta_0''$, and write β'' for $\beta_{\Gamma\cup\Lambda,\delta''}$. If $\gamma \in \mathcal{L}$ then $|\widehat{\beta''}(\gamma)| \geq 1/2$, so

$$\sum_{\gamma \in \widehat{G}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)||\widehat{\beta''}(\gamma)|^2 \geq 2^{-3}\alpha^2 + O(d\delta'\delta^{-1}). \tag{11}$$

Now, by Lemma 6.6 we have

$$\sum_{\gamma \in \widehat{G}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)||\widehat{\beta''}(\gamma)|^2 \leq \alpha^2 \max_{1\leq i\leq 2} \alpha_i^{-2}\langle f_i * \beta'', (f_i d\beta) * \beta''\rangle.$$

Which, combined with (11), ensures that there is some k with $1 \leq k \leq 2$ such that

$$\langle f_k * \beta'', (f_k d\beta) * \beta''\rangle \geq \alpha_k^2\left(2^{-3} + O(d\delta'\delta^{-1}\alpha^{-2})\right).$$

Now we can apply Lemma 5.2 to get that

$$\|\chi_{A_k} * \beta''\|_\infty \geq \alpha_k\left(1 + 2^{-3} + O(d\delta'\delta^{-1}\alpha^{-2})\right) + O(d\delta''\delta^{-1}).$$

However for $1 \leq i \leq 2$ we have

$$\begin{aligned}\|\chi_{A_i} * \beta''\|_\infty &\geq \|\chi_{A_i} * \beta''\|_{L^1(\beta)}\\ &= \int \chi_{A_i} d(\beta * \beta'')\\ &= \alpha_i + O(d\delta''\delta^{-1}) \text{ by Corollary 3.3 since } \operatorname{supp}\beta'' \subset B(\Gamma, \delta''),\end{aligned}$$

so that

$$\|\chi_{A_1} * \beta''\|_\infty\|\chi_{A_2} * \beta''\|_\infty \geq \alpha^2(1+2^{-3}) + O(d\delta'\delta^{-1}\alpha^{-2}) + O(d\delta''\delta^{-1}).$$

Assume that $\delta'' \leq \delta'$, so that

$$\|\chi_{A_1} * \beta''\|_\infty\|\chi_{A_2} * \beta''\|_\infty \geq \alpha^2(1+2^{-3}) + O(d\delta'\delta^{-1}\alpha^{-2}).$$

We now pick δ' regular for Γ such that the error term in the above expression is at most $2^{-4}\alpha^2$. This can be done by Proposition 3.1 whilst keeping $\delta' \gg \alpha^4\delta/d$. Finally we pick δ'' regular for $\Gamma\cup\Lambda$ subject to the two assumptions of $\delta'' \leq \delta'$ and $\delta'' \leq \delta_0''$. This can be done by Proposition 3.1 whilst keeping $\delta'' \gg \alpha^6\delta/d^3\log\sigma^{-1}$, and so the lemma is proved. □

We now prove the two technical claims we required.

Lemma 6.5. *Suppose that G is a compact Abelian group and $B(\Gamma, \delta)$ is a Bohr set. Suppose that $f_1, f_2 \in L^2(\beta)$ and $h \in \ell^\infty(\widehat{G})$. Then*

$$\sum_{\gamma \in \widehat{G}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)|h(\gamma) \leq \|f_1\|_{L^2(\beta)}\|f_2\|_{L^2(\beta)}\|h\|_{\ell^\infty(\widehat{G})}.$$

Proof. Start with the fact that

$$\sum_{\gamma\in\widehat{G}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)|h(\gamma) < \|h\|_{\ell^\infty(\widehat{G})} \sum_{\gamma\in\widehat{G}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)|.$$

We estimate the sum on the right using the Cauchy-Schwarz inequality and Plancherel's theorem.

$$\begin{aligned}\sum_{\gamma\in\widehat{G}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)| &\leq \left(\sum_{\gamma\in\widehat{G}} |\widehat{f_1}(\gamma)|^2\right)^{\frac{1}{2}} \left(\sum_{\gamma\in\widehat{G}} |\widehat{f_2 d\beta}(\gamma)|^2\right)^{\frac{1}{2}} \\ &= \|f_1\|_2 \|f_2 d\beta\|_2 \\ &= \|f_1\|_{L^2(\beta)} \|f_2\|_{L^2(\beta)} \text{ since } \beta \text{ is uniform on } B(\Gamma,\delta).\end{aligned}$$

Putting these two inequalities together gives the result. □

Similarly we have the following.

Lemma 6.6. *Suppose that G is a compact Abelian group and $B(\Gamma,\delta)$ and $B(\Gamma',\delta')$ are Bohr sets. Suppose that $f_1, f_2 \in L^2(\beta)$. Then*

$$\sum_{\gamma\in\widehat{G}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)||\beta'(\gamma)|^2 \leq \|f_1\|_{L^2(\beta)} \|f_2\|_{L^2(\beta)} \max_{1\leq i\leq 2} \|f_i\|^{-2}_{L^2(\beta)} \langle f_i * \beta', (f_i d\beta) * \beta'\rangle.$$

Proof. Apply the Cauchy-Schwarz inequality to the sum on the left to bound it above by

$$\left(\sum_{\gamma\in\widehat{G}} |\widehat{f_1}(\gamma)|^2 |\widehat{\beta'}(\gamma)|^2\right)^{\frac{1}{2}} \left(\sum_{\gamma\in\widehat{G}} |\widehat{f_2 d\beta}(\gamma)|^2 |\widehat{\beta'}(\gamma)|^2\right)^{\frac{1}{2}};$$

as before this can be rewritten as

$$\left(\sum_{\gamma\in\widehat{G}} \widehat{f_1}(\gamma)\widehat{\beta''}(\gamma)\overline{\widehat{f_1 d\beta}(\gamma)\widehat{\beta'}(\gamma)}\right)^{\frac{1}{2}} \left(\sum_{\gamma\in\widehat{G}} \widehat{f_2}(\gamma)\widehat{\beta''}(\gamma)\overline{\widehat{f_2 d\beta}(\gamma)\widehat{\beta'}(\gamma)}\right)^{\frac{1}{2}}.$$

Now apply Plancherel's theorem to this to conclude that it is equal to

$$\langle f_1 * \beta', (f_1 d\beta) * \beta'\rangle^{\frac{1}{2}} \langle f_2 * \beta', (f_2 d\beta) * \beta'\rangle^{\frac{1}{2}},$$

from which the lemma follows. □

The next lemma is a local version of the following easy application of the pigeonhole principle: If $A \subset \mathbb{Z}/N\mathbb{Z}$ has density at least $1-\sigma$ then A contains an arithmetic progression of length roughly σ^{-1}. It turns out not to be hard to localize this observation.

Lemma 6.7. *Suppose that $G = \mathbb{Z}/N\mathbb{Z}$ and $B(\Gamma,\delta)$ is a regular Bohr set. Suppose that $A \subset G$. Suppose that $\sigma \in (0,1]$. Suppose that A contains at least a proportion $1-\sigma$ of $B(\Gamma,\delta)$. Then either $\sigma^{-1} \gg d^{-1}\delta N^{\frac{1}{d}}$ or A contains an arithmetic progression of length at least $(4\sigma)^{-1}$.*

Proof. Let η be a constant to be optimized later.

First we find a large number of long arithmetic progressions in $B(\Gamma,\delta)$, all with the same common difference. Pick $y \neq 0$ from $B(\Gamma, 2^{\frac{1}{d}}N^{-\frac{1}{d}})$; such a y certainly exists by Lemma 2.1 which ensures that $|B(\Gamma, 2^{\frac{1}{d}}N^{-\frac{1}{d}})| \geq 2$. It follows that

$$x \in B(\Gamma, \delta(1-\eta)) \Rightarrow x, x+y, x+2y, ..., x+Ly \in B(\Gamma,\delta)$$

for $L \leq \eta\delta N^{\frac{1}{d}}2^{-\frac{1}{d}}$. Hence if $(4\sigma)^{-1} \leq \eta\delta N^{\frac{1}{d}}2^{-\frac{1}{d}}$ then there are at least $\mu_G(B(\Gamma,\delta(1-\eta)))N$ arithmetic progressions of common difference y and length $(4\sigma)^{-1}$ in $B(\Gamma,\delta)$. Moreover, since the common difference is the same for each progression, each point is in at most $2.(4\sigma)^{-1} = (2\sigma)^{-1}$ of these progressions.

If A does not contain any of these progressions then it misses at least one point in each progression and hence at least $\mu_G(B(\Gamma,\delta(1-\eta)))N/(2\sigma)^{-1}$ points of $B(\Gamma,\delta)$. It follows that

$$1-\sigma \leq \int \chi_A d\beta \leq 1 - 2\sigma\frac{\mu_G(B(\Gamma,\delta(1-\eta)))}{\mu_G(B(\Gamma,\delta))}.$$

By regularity of δ we can pick $\eta \gg d^{-1}$ such that

$$\frac{\mu_G(B(\Gamma,\delta(1-\eta)))}{\mu_G(B(\Gamma,\delta))} \geq \frac{2}{3},$$

from which we conclude that $1-\sigma \leq 1-4\sigma/3$, this contradicts the fact that σ is positive and so the lemma is proved. $\square$

Finally we put the previous two results together to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\sigma > 0$ be a constant to be optimized later. We construct a sequence of regular Bohr sets $B(\Gamma_k,\delta_k)$ iteratively. Write

$$\beta_k := \beta_{\Gamma_k,\delta_k}, d_k := |\Gamma_k| \text{ and } \alpha_k := \sqrt{\|\chi_{A_1} * \beta_k\|_\infty \|\chi_{A_2} * \beta_k\|_\infty}.$$

We initialize the iteration with $\Gamma_0 = \{0_{\widehat{G}}\}$ and $\delta_0 \gg 1$ regular for Γ_0 by Proposition 3.1.

Suppose that we are at stage k of the iteration. Since $B(\Gamma_k,\delta_k)$ has positive measure $\chi_{A_i} * \beta_k$ is continuous and hence we make take x_1 and x_2 such that

$$\chi_{A_i} * \beta_k(x_i) = \|\chi_{A_i} * \beta_k\|_\infty \text{ for } i = 1,2.$$

Apply Lemma 6.4 to the sets $(A_1 - x_1) \cap B(\Gamma_k,\delta_k)$ and $(A_2 - x_2) \cap B(\Gamma_k,\delta_k)$ and the regular Bohr set $B(\Gamma_k,\delta_k)$.

(1) Either $(A_1 - x_1) + (A_2 - x_2)$ contains at least a proportion $1-\sigma$ of some regular Bohr set $B(\Gamma_k,\delta_k')$ with $\delta_k' \gg \alpha_k^4\delta_k/d_k$. In which case we apply Lemma 6.7 to conclude that either $\sigma^{-1} \gg d_k^{-1}\delta_k' N^{\frac{1}{d_k}}$ or $A_1 + A_2 - x_1 - x_2$ (and hence $A_1 + A_2$) contains an arithmetic progression of length $(4\sigma)^{-1}$.
(2) Or there is a regular Bohr set $B(\Gamma_{k+1},\delta_{k+1})$ such that

$$\alpha_{k+1}^2 \geq \alpha_k^2(1+2^{-4}), \delta_{k+1} \gg \frac{\alpha_k^6\delta_k}{d_k^3\log\sigma^{-1}} \text{ and } d_{k+1} - d_k \ll \alpha_k^{-2}\log\sigma^{-1}.$$

From these last expressions we conclude that

$$\alpha_k^2 \geq \alpha^2(1+2^{-4})^k,$$

and hence, since $\alpha_k \leq 1$, the iteration terminates with $k \ll \log\alpha^{-1}$. It follows that

$$d_k \ll \sum_{k=0}^{\infty} \alpha_k^{-2}\log\sigma^{-2} \leq \alpha^{-2}\log\sigma^{-1}\sum_{k=0}^{\infty}(1+2^{-4})^{-k} \ll \alpha^{-2}\log\sigma^{-1},$$

and that

$$\delta_k \gg \left(\frac{\alpha}{\log\sigma^{-1}}\right)^{C\log\alpha^{-1}}$$

for some absolute constant $C > 0$.

For the iteration to terminate we must have arrived in the first case at some point, and hence either

$$\sigma^{-1} \gg \left(\frac{\alpha}{\log \sigma^{-1}}\right)^{C \log \alpha^{-1}} N^{c\alpha^2 (\log \sigma^{-1})^{-1}} \tag{12}$$

for some absolute constants $C, c > 0$ or there is an arithmetic progression in $A_1 + A_2$ of length $(4\sigma)^{-1}$. The result follows on taking σ^{-1} as large a possible whilst not satisfying (12). $\square$

7. Proof of Theorem 2.4

As before we begin with a brief overview of the argument in the finite-field setting, which can be made to prove the following result.

Theorem 7.1. *Suppose that G is a compact vector space over $\mathbb{F}_2$ and $A \subset G$ has density $\alpha > 0$. Then $A + A + A$ contains (up to a null set) an affine subspace of codimension at most $4\alpha^{-1}$.*

The proof is driven by the following iteration lemma.

Lemma 7.2. (Model iteration lemma) *Suppose that G is a compact vector space over $\mathbb{F}_2$. Suppose that $A \subset G$ has density α. Then at least one of the following is true.*

(1) *$A + A + A$ contains G (up to a null set).*
(2) *(Density increment) There is a subspace V of G such that*

$$\|\chi_A * \mu_V\|_\infty \geq \alpha(1 + \alpha/2) \text{ and } \operatorname{cod} V \leq 1.$$

The iteration lemma is not conceptually difficult; we sketch the main ideas now. Write $f := \chi_A * \chi_A * \chi_A$. If f is never zero (except for a null set) then $A + A + A$ certainly contains G (up to a null set), otherwise $f(x) = 0$ on a set of positive measure so there is a value of x for which

$$\sum_{\gamma \in \widehat{G}} \widehat{\chi_A}(\gamma)^3 \gamma(x) = f(x) = 0$$

by the inversion formula. Plancherel's theorem and the triangle inequality in the usual fashion give

$$\sum_{\gamma \neq 0_{\widehat{G}}} |\widehat{\chi_A}(\gamma)|^3 \geq \alpha^3,$$

from which it follows that there is a non-trivial characters γ at which $|\widehat{f}(\gamma)|$ is large. Such a bound provides an ℓ^∞ density increment for A.

Having proved this lemma the iteration is simple. Either $A + A + A$ contains a large affine subspace or we can increment the density of α. The density can't be incremented indefinitely and so eventually $A + A + A$ contains a large affine subspace.

To localize the iteration argument is not as easy as it appears. For the case $m = 3$ the argument is really just Bourgain's original argument for Roth's theorem. A particularly good exposition of this, due to Tao, can be found in [Tao04b]. There is a second exposition also due to Tao in [Tao04a] which uses smoothed measures in place of our βs. For the generalization to $m > 3$ the arguments in [Tao04b] appear insufficient; in particular the third claim in the proof below requires a new approach,

which it turns out was also used in [Tao04a]. However, this is all, perhaps, best illustrated by simply following the proof.

Lemma 7.3. (Iteration lemma) *Suppose that G is a compact Abelian group and $B(\Gamma,\delta)$ a regular Bohr set in G. Suppose that $A_1,...,A_m \subset B(\Gamma,\delta)$. Write α for the geometric mean of the densities of the sets $A_1,...,A_m$ in $B(\Gamma,\delta)$. Then at least one of the following is true.*

(1) *There is a δ' regular for Γ such that*
$$\delta' \gg \min_i\{\int \chi_{A_i} d\beta\}^2\delta/md$$
and $A_1+...+A_m$ contains a translate of $B(\Gamma,\delta')$ (up to a null set).

(2) *There is a set of characters Γ' and a δ'' regular for Γ' such that*
$$|\Gamma'| \leq |\Gamma|+1, \delta'' \gg \min_i\{\int \chi_{A_i} d\beta\}^3\delta/md^2$$
and
$$\left(\prod_{i=1}^m \|\chi_{A_i} * \beta_{\Gamma',\delta''}\|_\infty\right)^{\frac{1}{m}} \geq \alpha\left(1+\frac{\alpha^{\frac{1}{m-2}}}{2^8 m}\right).$$

As with Lemma 6.4 the proof which follows is rather complex with a number of sub-claims being necessary. To ease understanding we relegate proofs of these technical results to the end. The proof itself essentially splits up the situation into the various ways in which we can arrive at a density increment and then the technical lemmas deal provide the density increments in each case.

Proof. We may certainly assume that $\alpha > 0$ since otherwise we are done for trivial reasons. Let δ' be a constant, regular for Γ, to be chosen later. We may certainly assume that A_1 and A_2 have the largest densities on $B(\Gamma,\delta)$ and so it is $A_3,...,A_m$ we choose to move to the narrower Bohr neighborhood $B(\Gamma,\delta')$.
$$\int \chi_{A_i} * \beta' d\beta = \int \chi_{A_i} d(\beta * \beta') = \int \chi_{A_i} d\beta + O(d\delta'\delta^{-1}),$$
by Corollary 3.3. It follows by averaging that there is some $x_i \in B(\Gamma,\delta)$ such that
$$\chi_{A_i} * \beta'(x_i) \geq \int \chi_{A_i} d\beta + O(d\delta'\delta^{-1}). \tag{13}$$
Without loss of generality we may assume that the x_is are all zero. Write
$$\alpha_k := \int \chi_{A_k} d\beta \text{ and } f_k := (\chi_{A_k} - \alpha_k)\chi_B \text{ for } 1 \leq k \leq 2,$$
$$\text{and } \alpha_k := \int \chi_{A_k} d\beta' \text{ and } f_k := (\chi_{A_k} - \alpha_k)\chi_{B'} \text{ for } 3 \leq k \leq m.$$

Define
$$S := B' \setminus \operatorname{supp} \chi_{A_1}\chi_B * \chi_{A_2} d\beta * \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta'$$
and write σ for the density of S in B'. Now $A_1+...+A_m \supset \operatorname{supp} \chi_{A_1}\chi_B * \chi_{A_2} d\beta * \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta'$ so it follows that if $\sigma = 0$ then we are in the first case of the lemma. Hence we assume that $\sigma > 0$. We investigate the natural inner product
$$I := \langle f_1 * f_2 d\beta * f_3 d\beta' * ... * f_m d\beta', \chi_S\rangle_{L^2(\beta')}. \tag{14}$$

We can rewrite $f_1 * f_2 d\beta * f_3 d\beta' * ... * f_m d\beta'$ as

$$
\begin{aligned}
&\chi_{A_1}\chi_B * \chi_{A_2} d\beta * \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta' \\
&-\alpha_1 \chi_B * \chi_{A_2} d\beta * \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta' \\
&-\chi_{A_1}\chi_B * \alpha_2 \beta * \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta' \\
&+\alpha_1 \chi_B * \alpha_2 \beta * \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta' \\
&-f_1 * f_2 d\beta * \alpha_3 \beta' * \chi_{A_4} d\beta' * ... * \chi_{A_m} d\beta' \\
&-... \\
&-f_1 * f_2 d\beta * f_3 d\beta' * ... * f_{j-1} d\beta' * \alpha_j \beta' * \chi_{A_{j+1}} d\beta' * ... * \chi_{A_m} d\beta' \\
&-... \\
&-f_1 * f_2 d\beta * f_3 d\beta' * ... * f_{m-1} d\beta' * \alpha_m \beta'.
\end{aligned} \tag{15}
$$

There are three different types of term in this decomposition. The first term is unique and we denote it by Z, the next three are all of the same type and we denote them by T_1, T_2 and T_3. Finally the remaining terms are all of the same type and for $3 \le j \le m$ we write

$$S_j = -f_1 * f_2 d\beta * f_3 d\beta' * ... * f_{j-1} d\beta' * \alpha_j \beta' * \chi_{A_{j+1}} d\beta' * ... * \chi_{A_m} d\beta'.$$

We have

$$
\begin{aligned}
I = &\langle Z, \chi_S \rangle_{L^2(\beta')} + \langle T_1, \chi_S \rangle_{L^2(\beta')} + \langle T_2, \chi_S \rangle_{L^2(\beta')} + \langle T_3, \chi_S \rangle_{L^2(\beta')} \\
&+\langle S_3, \chi_S \rangle_{L^2(\beta')} + ... + \langle S_m, \chi_S \rangle_{L^2(\beta')}.
\end{aligned} \tag{16}
$$

Our objective now is to estimate the inner products on the right.

The first inner product is zero since χ_S is supported on the relative complement of Z. The inner products $\langle T_i, \chi_S \rangle_{L^2(\beta')}$ can all be estimated in the same way using the following claim which is Lemma 7.4.

Claim 2. *Suppose that $f \in L^\infty(\beta)$. Then*

$$\langle f * \beta * \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta', \chi_S \rangle_{L^2(\beta')} = \sigma \alpha_3 ... \alpha_m \times \left(\int f d\beta + O(md\delta' \delta^{-1} \|f\|_\infty) \right).$$

Note that the T_is can be rewritten as follows.

$$
\begin{aligned}
T_1 &= -\alpha_1 \chi_{A_2} \chi_B * \beta * \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta' \\
T_2 &= -\alpha_2 \chi_{A_1} \chi_B * \beta * \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta' \\
T_3 &= \alpha_1 \alpha_2 \chi_B * \beta * \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta'.
\end{aligned}
$$

Now apply the claim in each case to see that

$$
\begin{aligned}
\langle T_1, \chi_S \rangle_{L^2(\beta')} &= -\sigma \alpha_1 ... \alpha_m (1 + O(md\delta' \delta^{-1} \alpha_2^{-1})) \\
\langle T_2, \chi_S \rangle_{L^2(\beta')} &= -\sigma \alpha_1 ... \alpha_m (1 + O(md\delta' \delta^{-1} \alpha_1^{-1})) \\
\langle T_3, \chi_S \rangle_{L^2(\beta')} &= \sigma \alpha_1 ... \alpha_m (1 + O(md\delta' \delta^{-1})).
\end{aligned}
$$

It follows that

$$\langle T_1, \chi_S \rangle_{L^2(\beta')} + ... + \langle T_3, \chi_S \rangle_{L^2(\beta')} = -\sigma \alpha_1 ... \alpha_m (1 + O(md\delta' \delta^{-1} (\alpha_1^{-1} + \alpha_2^{-1}))),$$

and hence in (16) we have

$$I - \langle S_3, \chi_S \rangle_{L^2(\beta')} - ... - \langle S_m, \chi_S \rangle_{L^2(\beta')} = -\sigma \alpha_1 ... \alpha_m (1 + O(md\delta' \delta^{-1} (\alpha_1^{-1} + \alpha_2^{-1}))).$$

It follows that there is a $\delta_0' \gg \delta \min\{\alpha_1, \alpha_2\}/md$ such that if $\delta' \leq \delta_0'$ then the error term here is at most $1/2$. We assume that $\delta' \leq \delta_0'$ so that by the triangle inequality

$$|I| + |\langle S_3, \chi_S\rangle_{L^2(\beta')}| + ... + |\langle S_m, \chi_S\rangle_{L^2(\beta')}| \geq \frac{\sigma\alpha_1...\alpha_m}{2}.$$

It follows by averaging that one of the following is true.

$$|I| \geq \sigma\alpha_1...\alpha_m/4 \text{ or } |\langle S_j, \chi_S\rangle_{L^2(\beta')}| \geq \frac{\sigma\alpha_1...\alpha_m}{2^j} \text{ for some } 3 \leq j \leq m.$$

We have two claims which deal with the two cases: they are proved in Lemmas 7.5 and 7.6.

Claim 3. *If $|I| \geq \sigma\alpha_1...\alpha_m/4$ then there is a k with $3 \leq k \leq m$, a set of characters Γ' and a $\delta'' \leq \delta'$ regular for Γ' such that*

$$|\Gamma'| \leq |\Gamma| + 1, \delta'' \gg \frac{\alpha_k^2\delta'}{d} \textit{ and } \|\chi_{A_k} * \beta_{\Gamma',\delta''}\|_\infty \geq \alpha_k(1 + 2^{-5}\alpha^{\frac{1}{m-2}}).$$

Claim 4. *If*

$$|\langle S_j, \chi_S\rangle_{L^2(\beta')}| \geq \frac{\sigma\alpha_1...\alpha_m}{2^j}$$

then either

(1) *there is a k with $3 \leq k \leq j-1$, a set of characters Γ' and a $\delta'' \leq \delta'$ regular for Γ' such that*

$$|\Gamma'| \leq |\Gamma| + 1, \delta'' \gg \frac{\alpha_k\delta'}{d} \textit{ and } \|\chi_{A_k} * \beta_{\Gamma',\delta''}\|_\infty \geq \alpha_k(1 + 2^{-4});$$

(2) *or there is an k with $1 \leq k \leq 2$ and a $\delta'' \leq \delta'$ regular for Γ such that*

$$\delta'' \gg \min\{\frac{\alpha\delta'}{d}, \frac{\alpha_k\delta}{d}\} \textit{ and } \|\chi_{A_k} * \beta_{\Gamma,\delta''}\|_\infty \geq \alpha_k(1 + 2^{-7}).$$

Now it is just a matter of choosing δ' as large as possible whilst ensuring that the errors are small. From the claims we are guaranteed at least one of the following three outcomes.

(1) There is a k with $3 \leq k \leq m$, a set of characters Γ' and a $\delta'' \leq \delta'$ regular for Γ' such that

$$|\Gamma'| \leq |\Gamma| + 1, \delta'' \gg \frac{\alpha_k^2\delta'}{d} \text{ and } \|\chi_{A_k} * \beta_{\Gamma',\delta''}\|_\infty \geq \alpha_k(1 + 2^{-5}\alpha^{\frac{1}{m-2}}).$$

(2) There is a k with $3 \leq k \leq m-1$, a set of characters Γ' and a $\delta'' \leq \delta'$ regular for Γ' such that

$$|\Gamma'| \leq |\Gamma| + 1, \delta'' \gg \frac{\alpha_k\delta'}{d} \text{ and } \|\chi_{A_k} * \beta_{\Gamma',\delta''}\|_\infty \geq \alpha_k(1 + 2^{-4});$$

(3) There is a k with $1 \leq k \leq 2$ and a $\delta'' \leq \delta'$ regular for Γ such that

$$\delta'' \gg \min\{\frac{\alpha\delta'}{d}, \frac{\alpha_k\delta}{d}\} \text{ and } \|\chi_{A_k} * \beta_{\Gamma,\delta''}\|_\infty \geq \alpha_k(1 + 2^{-7}).$$

This imples that there is a k with $1 \leq k \leq m$, a set of characters Γ' and a δ'' regular for Γ' with

$$|\Gamma'| \leq |\Gamma| + 1 \text{ and } \delta'' \gg \frac{\min\{\alpha, \alpha_k^2\}\delta'}{d}$$

such that

$$\|\chi_{A_k} * \beta_{\Gamma',\delta''}\|_\infty \geq \alpha_k(1 + 2^{-7}\alpha^{\frac{1}{m-2}}).$$

Moreover

$$\begin{aligned}\|\chi_{A_i} * \beta_{\Gamma',\delta''}\|_\infty &\geq \|\chi_{A_i} * \beta_{\Gamma',\delta''}\|_{L^1(\beta)}\\ &= \int \chi_{A_i} d(\beta * \beta'')\\ &= \int \chi_{A_i} d\beta + O(d\delta''\delta^{-1}) \text{ by Corollary 3.3.}\end{aligned}$$

It follows that

$$\prod_{i=1}^m \|\chi_{A_k} * \beta_{\Gamma',\delta''}\|_\infty \geq (1 + 2^{-7}\alpha^{\frac{1}{m-2}}) \prod_{i=1}^m \left(\int \chi_{A_i} d\beta + O(d\delta''\delta^{-1})\right)$$

which in turn is at least

$$(1 + 2^{-7}\alpha^{\frac{1}{m-2}})\alpha^m . \prod_{i=1}^m \left(1 + O\left(d\delta''\delta^{-1}\left(\int \chi_{A_i} d\beta\right)^{-1}\right)\right).$$

When we take mth roots the product can be estimated by

$$\left(\prod_{i=1}^m \left(1 + O\left(d\delta''\delta^{-1} \int \chi_{A_i} d\beta^{-1}\right)\right)\right)^{\frac{1}{m}} = 1 + O(d\delta''\delta^{-1} \min_i \{\int \chi_{A_i} d\beta\}^{-1}),$$

so there is a δ_0'''

$$\delta_0''' \gg \frac{\delta \min_i \{\int \chi_{A_i} d\beta\} \alpha^{\frac{1}{m-2}}}{d}$$

such that if $\delta'' \leq \delta_0'''$ then

$$\prod_{i=1}^m \|\chi_{A_k} * \beta_{\Gamma',\delta''}\|_\infty^{\frac{1}{m}} \geq \left(1 + \frac{\alpha^{\frac{1}{m-2}}}{2^8 m}\right)\alpha.$$

Since $\delta'' \leq \delta'$ the conclusion of the lemma follows on taking δ' (regular by Proposition 3.1) as large as possible subject to $\delta' \leq \delta_0'$ and $\delta' \leq \delta_0'''$. $\square$

We now address the technical lemma which we employed above.

Lemma 7.4. *Suppose that G is a compact Abelian group, $B(\Gamma, \delta)$ is a regular Bohr set, $B(\Gamma, \delta')$ is a Bohr set, $f \in L^\infty(\beta)$ and $A_1, ..., A_m, S \subset B(\Gamma, \delta')$ are sets with relative density $\alpha_1, ..., \alpha_m$ and σ respectively. Then*

$$\langle f * \beta * \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta', \chi_S\rangle_{L^2(\beta')} = \sigma\alpha_3...\alpha_m \times \left(\int f d\beta + O(md\delta'\delta^{-1}\|f\|_\infty)\right).$$

Proof. We show that if $x \in B'$ then $f * \beta * \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta'(x)$ is a constant plus a small L^∞-error. This leads directly to the desired conclusion.

By Corollary 3.3 with $\mu = \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta'$ we have

$$\|\beta * \mu - \alpha_3...\alpha_m \beta\| = O(md\delta'\delta^{-1}\alpha_3...\alpha_m),$$

since $\operatorname{supp} \mu \subset B(\Gamma, m\delta')$. It follows that

$$\begin{aligned} f * \beta * \chi_{A_3} d\beta_3 * ... * \chi_{A_m} d\beta_m &= \alpha_3...\alpha_m f * \beta + O(\|f\|_\infty \|\beta * \mu - \alpha_3...\alpha_m\beta\|)\\ &= \alpha_3...\alpha_m f * \beta + O(md\delta'\delta^{-1}\|f\|_\infty \alpha_3...\alpha_m).\end{aligned}$$

If $x \in B'$ then by Corollary 3.4

$$f * \beta(x) = \int f d\beta + O(\|f\|_\infty d\delta'\delta^{-1}).$$

Combining these last two expressions we get

$$f * \beta * \chi_{A_3} d\beta' * ... * \chi_{A_m} d\beta'(x) = \int f d\beta \alpha_3 ... \alpha_m + O(\|f\|_\infty m d\delta'\delta^{-1}\alpha_3...\alpha_m).$$

The required estimate follows. □

Lemma 7.5. *Suppose that G is a compact Abelian group, $B(\Gamma, \delta)$ and $B(\Gamma, \delta')$ are regular Bohr sets, $A_1, A_2 \subset B(\Gamma, \delta)$ have relative density α_1 and α_2 respectively, and $A_3, ..., A_m, S \subset B(\Gamma, \delta')$ have relative density $\alpha_3, ..., \alpha_m$ and $\sigma > 0$ respectively. Write $f_i := (\chi_{A_i} - \alpha_i)\chi_B$ for $1 \leq i \leq 2$ and $f_i := (\chi_{A_i} - \alpha_i)\chi_{B'}$ for $3 \leq i \leq m$. If*

$$|\langle f_1 * f_2 d\beta * f_3 d\beta' * ... * f_m d\beta', \chi_S\rangle_{L^2(\beta')}| \geq \sigma\alpha_1...\alpha_m/4$$

then there is a k with $3 \leq k \leq m$, a set of characters Γ' and a $\delta'' \leq \delta'$ regular for Γ' such that

$$|\Gamma'| \leq |\Gamma| + 1, \delta'' \gg \frac{\alpha_k^2 \delta'}{d} \text{ and } \|\chi_{A_k} * \beta_{\Gamma', \delta''}\|_\infty \geq \alpha_k(1 + 2^{-5}\alpha^{\frac{1}{m-2}}).$$

Proof. Write $I := \langle f_1 * f_2 d\beta * f_3 d\beta' * ... * f_m d\beta', \chi_S\rangle_{L^2(\beta')}$. Plancherel's theorem tells us that

$$I = \sum_{\gamma \in \widehat{G}} \widehat{f_1}(\gamma)\widehat{f_2 d\beta}(\gamma)\widehat{f_3 d\beta'}(\gamma)...\widehat{f_m d\beta'}(\gamma)\overline{\widehat{\chi_S d\beta'}(\gamma)}.$$

Recalling that $|\widehat{\chi_S d\beta'}(\gamma)| \leq \|\chi_S\|_{L^1(\beta')} = \sigma$, we may apply the triangle inequality to get

$$\sigma \sum_{\gamma \in \widehat{G}} |\widehat{f_1}(\gamma)\widehat{f_2 d\beta}(\gamma)\widehat{f_3 d\beta'}(\gamma)...\widehat{f_m d\beta'}(\gamma)| \geq \frac{\sigma\alpha_1...\alpha_m}{4}.$$

Divide by σ (which is possible since $\sigma > 0$) to get

$$\sum_{\gamma \in \widehat{G}} |\widehat{f_1}(\gamma)\widehat{f_2 d\beta}(\gamma)\widehat{f_3 d\beta'}(\gamma)...\widehat{f_m d\beta'}(\gamma)| \geq \frac{\alpha_1...\alpha_m}{4}. \tag{17}$$

By the Cauchy-Schwarz inequality and Plancherel's theorem we have

$$\begin{aligned}
\sum_{\gamma \notin \mathcal{L}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)| &\leq \left(\sum_{\gamma \in \widehat{G}} |\widehat{f_1}(\gamma)|^2\right)^{\frac{1}{2}} \left(\sum_{\gamma \in \widehat{G}} |\widehat{f_2 d\beta}(\gamma)|^2\right)^{\frac{1}{2}} \\
&= \|f_1\|_2 \|f_2 d\beta\|_2 \\
&= \|f_1\|_{L^2(\beta)}\|f_2\|_{L^2(\beta)} \text{ since } \beta \text{ is uniform on } B, \\
&= (\alpha_1(1-\alpha_1)\alpha_2(1-\alpha_2))^{\frac{1}{2}} \leq \sqrt{\alpha_1\alpha_2} = \alpha,
\end{aligned}$$

so applying the triangle inequality to (17) we conclude that

$$\sup_{\gamma \in \widehat{G}} |\widehat{f_3 d\beta'}(\gamma)|...|\widehat{f_m d\beta'}(\gamma)| \geq \frac{\alpha\alpha_3...\alpha_m}{4}.$$

It follows that for some k with $3 \leq k \leq m$ we have

$$\sup_{\gamma \in \widehat{G}} |\widehat{f_k d\beta'}(\gamma)|^{m-2} \geq \frac{\alpha_k^{m-2}\alpha}{4}.$$

Apply Lemma 5.1 to get the lemma. □

Lemma 7.6. *Suppose that G is a compact Abelian group, $B(\Gamma,\delta)$ and $B(\Gamma,\delta')$ are regular Bohr sets, $A_1, A_2 \subset B(\Gamma,\delta)$ have relative density α_1 and α_2 respectively, and $A_3,...,A_m, S \subset B(\Gamma,\delta')$ have relative density $\alpha_3,...,\alpha_m$ and $\sigma > 0$ respectively. Write $f_i := (\chi_{A_i} - \alpha_i)\chi_B$ for $1 \le i \le 2$, $f_i := (\chi_{A_i} - \alpha_i)\chi_{B'}$ for $3 \le i \le m$, and*

$$S_j := -f_1 * f_2 d\beta * f_3 d\beta' * ... * f_{j-1} d\beta' * \alpha_j \beta' * \chi_{A_{j+1}} d\beta' * ... * \chi_{A_m} d\beta'$$

for $3 \le j \le m$. If

$$|\langle S_j, \chi_S\rangle_{L^2(\beta')}| \ge \frac{\sigma\alpha_1...\alpha_m}{2^j}$$

then either

1. *there is a k with $3 \le k \le j-1$, a set of characters Γ' and a $\delta'' \le \delta'$ regular for Γ' such that*
$$|\Gamma'| \le |\Gamma| + 1, \delta'' \gg \frac{\alpha_k \delta'}{d} \text{ and } \|\chi_{A_k} * \beta_{\Gamma',\delta''}\|_\infty \ge \alpha_k(1 + 2^{-4});$$
2. *or there is an k with $1 \le k \le 2$ and a $\delta'' \le \delta'$ regular for Γ such that*
$$\delta'' \gg \min\{\frac{\alpha\delta'}{d}, \frac{\alpha_k\delta}{d}\} \text{ and } \|\chi_{A_k} * \beta_{\Gamma,\delta''}\|_\infty \ge \alpha_k(1 + 2^{-7}).$$

Proof. Plancherel's theorem gives

$$\langle S_j, \chi_S\rangle_{L^2(\beta')} = \sum_{\gamma \in \widehat{G}} \widehat{S_j}(\gamma)\overline{\widehat{\chi_S d\beta'}(\gamma)}.$$

Recalling that $|\widehat{\chi_S d\beta'}(\gamma)| \le \|\chi_S\|_{L^1(\beta')} = \sigma$, we may apply the triangle inequality to get

$$|\langle S_j, \chi_S\rangle_{L^2(\beta')}| \le \sigma \sum_{\gamma \in \widehat{G}} |\widehat{S_j}(\gamma)|.$$

If we now use the assumption on the magnitude of the inner product and divide by σ (which is possible since $\sigma > 0$) we get

$$2^{-j}\alpha_1...\alpha_m \le \sum_{\gamma \in \widehat{G}} \left\{ \begin{array}{c} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)||\widehat{f_3 d\beta'}(\gamma)|...|\widehat{f_{j-1} d\beta'}(\gamma)| \\ \times|\alpha_j \widehat{\beta'}(\gamma)||\widehat{\chi_{A_{j+1}} d\beta'}(\gamma)|...|\widehat{\chi_{A_m} d\beta'}(\gamma)| \end{array} \right\}. \tag{18}$$

First we note that $|\widehat{\chi_{A_k} d\beta'}(\gamma)| \le \alpha_k$ for $j+1 \le k \le m$. Second if there is some k with $3 \le k \le j-1$ such that $|\widehat{f_k d\beta'}(\gamma)| \ge \alpha_k/2$ then we may apply Lemma 5.1 to get the density increment in the first case of the conclusion of the lemma. Hence we assume that $|\widehat{f_k d\beta'}(\gamma)| \le \alpha_k/2$ for all k with $3 \le k \le j-1$. These two observations serve to tell us that each summand in (18) is bounded above by

$$|\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)||\widehat{\beta'}(\gamma)|2^{-(j-3)}\alpha_3...\alpha_m.$$

Hence

$$2^{-3}\alpha_1\alpha_2 \le \sum_{\gamma \in \widehat{G}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)||\widehat{\beta'}(\gamma)|. \tag{19}$$

The characters at which $|\widehat{\beta'}(\gamma)|$ is large make a significant contribution to this sum, which we can see as follows. Write

$$\mathcal{L} := \{\gamma \in \widehat{G} : |\widehat{\beta'}(\gamma)| \ge 2^{-4}\sqrt{\alpha_1\alpha_2}\}.$$

Then

$$\sum_{\gamma \notin \mathcal{L}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)||\widehat{\beta'}(\gamma)| \leq 2^{-4}\sqrt{\alpha_1\alpha_2} \sum_{\gamma \in \widehat{G}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)|. \tag{20}$$

Now by the Cauchy-Schwarz inequality and Plancherel's theorem we have

$$\begin{aligned}
\sum_{\gamma \notin \mathcal{L}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)| &\leq \left(\sum_{\gamma \in \widehat{G}} |\widehat{f_1}(\gamma)|^2\right)^{\frac{1}{2}} \left(\sum_{\gamma \in \widehat{G}} |\widehat{f_2 d\beta}(\gamma)|^2\right)^{\frac{1}{2}} \\
&= \|f_1\|_2 \|f_2 d\beta\|_2 \\
&= \|f_1\|_{L^2(\beta)} \|f_2\|_{L^2(\beta)} \text{ since } \beta \text{ is uniform on } B, \\
&= (\alpha_1(1-\alpha_1)\alpha_2(1-\alpha_2))^{\frac{1}{2}} \leq \sqrt{\alpha_1\alpha_2}.
\end{aligned}$$

We can use this in (20) to see that

$$\sum_{\gamma \notin \mathcal{L}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)||\widehat{\beta'}(\gamma)| \leq 2^{-4}\alpha_1\alpha_2$$

and hence by (19) that

$$\sum_{\gamma \in \mathcal{L}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)||\widehat{\beta'}(\gamma)| \geq 2^{-4}\alpha_1\alpha_2.$$

Apply Lemma 3.5 to get a $\delta_0'' \gg \sqrt{\alpha_1\alpha_2}\delta'/d$ such that for all $\delta'' \leq \delta_0''$

$$\begin{aligned}
\mathcal{L} &\subset \{\gamma \in \widehat{G} : |1-\gamma(x)| \leq 1/2 \text{ for all } x \in B(\Gamma, \delta'')\} \\
&\subset \{\gamma \in \widehat{G} : |\widehat{\beta_{\Gamma,\delta''}}(\gamma)| \geq 1/2\}.
\end{aligned}$$

Write β'' for $\beta_{\Gamma,\delta''}$. Now, if $\gamma \in \mathcal{L}$ we have

$$|\widehat{\beta''}(\gamma)|^2 \geq |\widehat{\beta'}(\gamma)|/4,$$

so

$$\sum_{\gamma \in \widehat{G}} |\widehat{f_1}(\gamma)||\widehat{f_2 d\beta}(\gamma)||\widehat{\beta''}(\gamma)|^2 \geq 2^{-6}\alpha_1\alpha_2.$$

Now applying the Cauchy-Schwarz inequality and Plancherel's theorem as before, we get that the sum on the left bounded above by

$$\begin{aligned}
&\left(\sum_{\gamma \in \widehat{G}} |\widehat{f_1}(\gamma)\widehat{\beta''}(\gamma)|^2\right)^{\frac{1}{2}} \left(\sum_{\gamma \in \widehat{G}} |\widehat{f_2 d\beta}(\gamma)\widehat{\beta''}(\gamma)|^2\right)^{\frac{1}{2}} \\
&= \|f_1 * \beta''\|_2 \|(f_2 d\beta) * \beta''\|_2 \\
&= \langle f_1 * \beta'', (f_1 d\beta) * \beta''\rangle^{\frac{1}{2}} \langle f_2 * \beta'', (f_2 d\beta) * \beta''\rangle^{\frac{1}{2}} \text{ since } \beta \text{ is uniform on } B(\Gamma, \delta).
\end{aligned}$$

Hence

$$\langle f_1 * \beta'', (f_1 d\beta) * \beta''\rangle^{\frac{1}{2}} \langle f_2 * \beta'', (f_2 d\beta) * \beta''\rangle^{\frac{1}{2}} \geq 2^{-6}\alpha_1\alpha_2.$$

It follows that there is some k with $1 \leq k \leq 2$ such that

$$\langle f_k * \beta'', (f_k d\beta) * \beta''\rangle \geq 2^{-6}\alpha_k^2,$$

and applying Lemma 5.2 we get

$$\|\chi_{A_k} * \beta''\|_\infty \geq \alpha_k(1+2^{-6}) + O(d\delta''\delta^{-1}).$$

It follows from Proposition 3.1 that there is a choice of δ'' regular for Γ such that

$$\min\{\delta_0'', \delta'\} \geq \delta'' \gg \min\{\delta_0'', \delta\alpha_k/d\} \text{ and } \|\chi_{A_k} * \beta''\|_\infty \geq \alpha_k(1+2^{-7}).$$

This gives the second conclusion of the lemma once we note that $\alpha_1\alpha_2 \geq \alpha^2$. □

It is a simple matter to, as before, iterate this lemma.

Proof of Theorem 2.4. We construct a sequence of regular Bohr sets $B(\Gamma_k, \delta_k)$ iteratively. Write

$$\beta_k = \beta_{\Gamma_k,\delta_k}, d_k := |\Gamma_k| \text{ and } \alpha_k = \left(\prod_{i=1}^m \|\chi_{A_i} * \beta_k\|_\infty\right)^{\frac{1}{m}}. \tag{21}$$

We initialize the iteration with $\Gamma_0 = \{0_{\widehat{G}}\}$ and $\delta_0 \gg 1$ regular for Γ_0 by Proposition 3.1.

Suppose that we are at stage k of the iteration. Since $B(\Gamma_k, \delta_k)$ has positive measure $\chi_{A_i} * \beta_k$ is continuous and hence we make take $x_1, ..., x_m$ such that

$$\chi_{A_i} * \beta_k(x_i) = \|\chi_{A_i} * \beta_k\|_\infty.$$

Now we apply the iteration lemma to the sets $(A_1 - x_1) \cap B(\Gamma_k, \delta_k), (A_2 - x_2) \cap B(\Gamma_k, \delta_k), ..., (A_m - x_m) \cap B(\Gamma_k, \delta_k)$ and the regular Bohr set $B(\Gamma_k, \delta_k)$.

(1) Either $A_1 + ... + A_m$ contains (up to a null set) a translate of a Bohr set $B(\Gamma_k, \delta_k')$ with $\delta_k' \gg \alpha_k^{2m}\delta_k/md_k$.
(2) Or there is a regular Bohr set $B(\Gamma_{k+1}, \delta_{k+1})$ such that

$$\alpha_{k+1} \geq \alpha_k\left(1 + \frac{\alpha_k^{\frac{1}{m-2}}}{2^8 m}\right), \delta_{k+1} \gg \frac{\alpha_k^{3m}\delta_k}{md_k^2} \text{ and } d_{k+1} - d_k \leq 1.$$

From these last expressions we conclude that after at most $2^8 m\alpha_k^{-\frac{1}{m-2}}$ iterations the density doubles and so the iteration terminates and moreover

$$d_k \leq \sum_{j=0}^{\log_2 \alpha^{-1}} 2^8 m(2^j\alpha)^{-\frac{1}{m-2}} \leq \sum_{j=0}^{\infty} 2^8 m(2^j\alpha)^{-\frac{1}{m-2}} \ll m^2\alpha^{-\frac{1}{m-2}},$$

and hence

$$\delta_k \gg (c\alpha)^{Cm^3\alpha^{-\frac{1}{m-2}}}$$

for some absolute constants $C, c > 0$.

For the iteration to terminate we must have arrived at the first case at some point and the conclusion follows. □

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